SQUARE SUM DEGREE DIVIDED BY DIAMETER ENERGY OF COMPLEMENT GRAPH

Manasa H S[1], Kavita Permi[2*]

[1]Department of Mathematics, M.R.I.T, Mandya- 570028, India. E-mail:manasasangeetha09@gmail.com
[2*]Department of Mathematics, Presidency University, Bengaluru- 560064, India.
E-mail: kavitapermi@presidencyuniversity.in

ABSTRACT
The energy of graph was presented by Gutman [2] as the amount of the absolute values of the eigenvalues of the adjacency matrix of the given graph Γ. In this paper we find square sum degree divided by diameter matrix of complement graphs and also find their energy.
2010 Mathematics Subject Classification: 05C50

KEYWORDS: square sum degree, diameter, eigen values, spectrum and comple-ments.

1. INTRODUCTION

Let G be a simple graph with p vertices and let A be its contiguousness lat-tice. The characteristic polynomial of G is characterized a $\psi(G, x) = det(xI - A)$,,where I is the identity lattice of size p. The eigenvalues of G (for example the eigenvalues of A) are the roots $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_p$ of $\psi(G, x)$,all of which are real. The energy of G is defined to be The multiplicity of an eigenvalue λ of G is the order of λ as a root of $\psi(G, x)$.The spectrum of G is the array

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_p \\ n_1 & n_2 & \dots & n_p \end{pmatrix}.$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ are the distinct eigenvalues of G and n_i is the multiplicity of $\lambda_i (i = 1, 2, \ldots, p)$. It is easy to see that $n_1 + n_2 + \ldots + n_p$ On the other hand, because tr(A) = 0, we have $n1\lambda1 + n2\lambda2 + \ldots np\lambda p = 0$

It is well-known that if G is a k-regular graph, then k is an eigenvalue of G with multiplicity c, where c is the number of connected components of G, and that the absolute value of each eigenvalue of G is less than or equal to k

The energy of a graph G, denoted by E(G), is defined as the sum of the absolute values of the eigenvalues of G. that is

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

This concept was introduced by Gutman in 1978. The energy of a graph is a quantity closely related to the Huckel molecular orbital total Π- electron energy [3]. For more about the energy of graph refer [4], [5], [6], [7], [8], [9], [10], [11], [12].

2. Square sum degree divided by diameter ma-trix and its complement energy

Let G = (V, E) be a connected simple graph with $|V| = p$ vertices and |E| = m edges and let di denote the degree of the vertex vi , $for\ i = 1, 2, \ldots\ldots, p$. For ver-tices $vi, vj \in V(G)$,the distance $d(v_i, v_j)$is defined as the length of a shortest path between v_i and v_j in G . The eccentricity of a vertex is the maximum distance from it to any other vertex, $\varepsilon_i = maxv_j \in V(G) d(v_i, v_j)$.

The diameter of a graph diam(G) is the maximum eccentricity of any vertex in the graph, or the greatest distance between any pair of vertices.

Let d_i, d_j be the degrees of the vertices v_i, v_j respectively for all $i, j = 1, 2, 3, \ldots, p$ of the complement graph G then the square sum degree by di-ameter matrix of of the complement graph is defined as

$$x_{ij} = \begin{cases} \frac{d_i^2 + d_j^2}{diam(\overline{G})} & \text{if there is a path between } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}$$

The square sum degree divided by diameter matrix is a symmetric matrix with eigen values as $\psi 1 \geq \psi 2 \geq \psi 3 \geq \ldots \geq \psi p$ The characteristic polynomial of $diam\ SSD\ (G)$ is given by $det\left|\psi I - \frac{SSD}{diam}(\bar{G})\right|$. The square sum degree divided by diameter energy of the complement graph G is defined as sum of absolute values of ζi , $i = 1,2, \ldots, p$

$$E\left[\frac{SD}{diam}(\overline{G})\right] = \sum_{i=1}^{p} |psi_i|$$

3. Properties of square sum degree divided by diameter energy of complement graphs

Theorem 3.1. If eigen values of $\frac{SSD}{diam}(\bar{G})$ $are\ \psi 1 + > \psi 2 + > \cdots > \psi r$, then

$$1.\ \sum_{i=1}^{p} \psi_i = 0\ \ and$$

$$2.\ \sum_{i=1}^{p} \psi_i^2 = 2\sum_{i=1}^{p}\left(\frac{d_i^2 + d_j^2}{diam(\overline{G})}\right)^2 = 2\Phi$$

$$where\ \Phi = \sum_{i=1}^{p}\left(\frac{d_i^2 + d_j^2}{diam(\overline{G})}\right)^2.$$

Proof. (1) Since the diagonal entries are zero the sum of leading diagonal entries of $\frac{SSD}{diam}(\bar{G})$ is zero .

$$\text{Hence} \sum_{i=1}^{p} \psi_i = 0.$$

(2) The sum of squares of latent roots of $\frac{SSD}{diam}(G)$is the sum of latent roots of $\left[\frac{SSD}{diam}(\bar{G})\right]^2$,

$$\sum_{i=1}^{p} \psi_i^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} u_{ij}\, u_{ji}$$
$$= 0 + 2\sum_{i<j} (u_{ij})^2$$
$$= 2\sum_{i=1}^{p}\left(\frac{d_i^2 + d_j^2}{diam(\overline{G})}\right)^2$$
$$= 2\Phi.$$

Theorem 3.2. If c_0, c_1 and c_2 are the first three coefficients of characteristic polynomial of $\frac{SSD}{diam}(\bar{G})$ matrix, then

1. $c_0 = 1,$
2. $c_1 = 0\ and$
3. $c_2 = -\Phi$.

Proof. (i)By definition, $\Gamma(\psi, x) = det[\psi I - \Phi]$
Therefore $c_0 = 1.$
(ii)$c_1 = (-1)1 \times trace(\Gamma) = -1 \times 0 = 0.$
(iii) $By\ definition\ c_2 =$

$$\sum_{1 \leq i < j \leq p} \begin{vmatrix} u_{ii} & u_{ij} \\ u_{ji} & u_{jj} \end{vmatrix} = \sum_{1 \leq i < j \leq p} (a_{ii}u_{jj} - u_{ij}u_{ji})$$

$$= \sum_{1\le i<j\le p} u_{ii}u_{jj} - \sum_{1\le i<j\le p} a_{ij}^2 = 0 - \Phi = -\Phi\ .$$

We have the following bounds for $\frac{SSD}{diam}(\bar{G})$using McClelland's inequalities.

Theorem 3.3. *Let G* be a complement graph with p vertices, then the upper bound for $\frac{SSD}{diam}(\overline{G})$ is

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \le \sqrt{2p\Phi}.$$

Proof. Let $\psi 1 \ge \psi 2 \ge \cdots \ge \psi p$ be the eigen values of $\frac{SSD}{diam}(\overline{G})$, then by Using Cauchy-Schwarz inequality we have,

$$\left[\sum_{i=1}^{p} u_i v_i\right]^2 \le \left[\sum_{i=1}^{p} u_i^2\right]\left[\sum_{i=1}^{p} v_i^2\right].$$

Choose $ui = 1,\ vi = \psi i$ and by Theorem 3.1

$$\left[\sum_{i=1}^{p} |\psi_i^+|\right]^2 \le \left[\sum_{i=1}^{p} 1\right]\left[\sum_{i=1}^{p} |\psi_i^+|^2\right] = p\sum_{i=1}^{p} {\psi_i^+}^2$$

$$\left[\left(\frac{SSD}{diam}\right)E(\overline{G})\right]^2 \le p2\Phi.$$

Hence

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \le \sqrt{2p\Phi}.$$

We present the following lower bounds for E $\frac{SSD}{diam}(\bar{G})$

Theorem 3.4. Let G be a graph with p vertices. If $\tau = \left|det\frac{SSD}{diam}(\overline{G})\right|\ of\ \overline{G},$ then the lower bound is

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \ge \sqrt{2\Phi + p(p-1)\tau^{\frac{2}{p}}}.$$

Proof. By definition we have,

$$\left[E\left(\frac{SSD}{diam}(\overline{G})\right)\right]^2 = \left[\sum_{i=1}^{p} |\psi_i|\right]^2 = \left[\sum_{i=1}^{p} |\psi_i|\right]\left[\sum_{j=1}^{p} |\psi_j|\right]$$

$$= \sum_{i=1}^{p} |\psi_i|^2 + \sum_{i\ne j} |\psi_i||\psi_j|.$$

From the inequality of arthimetic and geometric means

$$\frac{1}{p(p-1)}\sum_{i\neq j}|\psi_i||\psi_j| \geq \left[\prod_{i\neq j}|\psi_i||\psi_j|\right]^{\frac{1}{p(p-1)}}.$$

Therefore

$$\left[E\left(\frac{SSD}{diam}(\overline{G})\right)\right]^2 \geq \sum_{i=1}^{p}|\psi_i|^2 + p(p-1)\left[\prod_{i\neq j}|\psi_i||\psi_j|\right]^{\frac{1}{p(p-1)}}$$

$$\geq \sum_{i=1}^{p}|\psi_i|^2 + p(p-1)\left[\prod_{i=1}^{p}|\psi_i|^{2(p-1)}\right]^{\frac{1}{p(p-1)}}.$$

$$= \sum_{i=1}^{p}|\psi_i|^2 + p(p-1)\left|\prod_{i=1}^{p}\psi_i\right|^{\frac{2}{p}}$$

$$= 2\Phi + p(p-1)\tau^{\frac{2}{p}}.$$

Hence

$$E\left[\frac{SSD}{diam}(\overline{G}))\right] \geq \sqrt{2\Phi + p(p-1)\tau^{\frac{2}{p}}}.$$

Theorem 3.5. Let r_i *and* $s_i, 1 \leq i \leq p$ be positive real numbers with $M_1 = max_{\{1\leq i\leq p\}}(r_i)$, $M_2 = max_{\{1\leq i\leq p\}}(s_i), m_1 = min_{\{1\leq i\leq p\}}(r_i), m_2 = min_{\{1\leq i\leq n\}}(s_i)$ then by theorem 90 of [12]

$$\sum_{i=1}^{p}r_i^2\sum_{i=1}^{p}s_i^2 \leq \frac{1}{4}\left[\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right]^2\left[\sum_{i=1}^{n}r_is_i\right]^2.$$

Theorem 3.6. For a graph G with p vertices *let* $|\psi_1|$ *and* $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi i|'$s of $\frac{SSD}{diam}(\bar{G})$respectively, then we have

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \geq \frac{\sqrt{8p\Phi|\psi_1||\psi_p|}}{|\psi_1|+|\psi_p|}.$$

Proof. Consider a graph G with p vertices let $|\psi_1|$ *and* $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi_i|'s$ of $\frac{SSD}{diam}(\bar{G})$respectively.

From theorem 3.5,

$$\sum_{i=1}^{p}r_i^2\sum_{i=1}^{p}s_i^2 \leq \frac{1}{4}\left[\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right]^2\left[\sum_{i=1}^{p}r_is_i\right]^2.$$

Let $r_i = 1, s_i = |\zeta_i|, M_1M_2 = |\psi_i|, m_1m_2 = |\psi p|$ then

$$\sum_{i=1}^{p} 1^2 \sum_{i=1}^{p} \psi_i^2 \leq \frac{1}{4}\left[\sqrt{\frac{|\psi_1|}{|\psi_p|}}+\sqrt{\frac{|\psi_n|}{|\psi_1|}}\right]^2 \left[\sum_{i=1}^{p} 1|\psi_i|\right]^2$$

From theorem 3.1

$$p2\Phi \leq \frac{1}{4}\left[\frac{(|\psi_1|+|\psi_p|)^2}{|\psi_1||\psi_n|}\right]\left[E\left(\frac{SSD}{diam}(\overline{G})\right)\right]^2,$$

$$\left[E\left(\frac{SSD}{diam}(\overline{G})\right)\right]^2 \geq \frac{8p\Phi|\psi_1||\psi_p|}{(|\psi_1|+|\psi_p|)^2},$$

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \geq \frac{\sqrt{8p\Phi|\psi_1||\psi_p|}}{|\psi_1|+|\psi_p|}.$$

Theorem 3.7. Let r_i *and* $s_i, 1 \leq i \leq p$ be non negative real numbers with $M_1 = max_{\{1\leq i\leq p\}}(r_i)$, $M_2 = max_{\{1\leq i\leq p\}}(s_i), m_1 = min_{\{1\leq i\leq p\}}(r_i), m_2 = min_{\{1\leq i\leq n\}}(s_i)$ then by theorem 3.1 of

$$\sum_{i=1}^{p} r_i^2 \sum_{i=1}^{p} s_i^2 - \left[\sum_{i=1}^{p} r_i s_i\right]^2 \leq \frac{p^2}{4}(M_1M_2 - m_1m_2)^2.$$

Theorem 3.8. For a graph G with p vertices, we have

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \geq \sqrt{2p\Phi - \frac{p^2}{4}(|\psi_1|-|\psi_p|)^2}.$$

Proof. Consider a graph G with p vertices let $|\psi_1|$ *and* $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi_i|$'s of $\frac{SSD}{diam}(\bar{G})$ respectively. From theorem 3.7,

$$\sum_{i=1}^{p} r_i^2 \sum_{i=1}^{p} s_i^2 - \left[\sum_{i=1}^{p} r_i s_i\right]^2 \leq \frac{p^2}{4}(M_1M_2 - m_1m_2)^2.$$

Let $r_i = 1, s_i = |\psi_i|, M_1 M_2 = |\psi_i|, m_1 m_2 = |\psi_p|$, then

$$\sum_{i=1}^{p} 1^2 \sum_{i=1}^{p} \psi_i^2 - \left[\sum_{i=1}^{p} 1|\psi_i|\right]^2 \leq \frac{p^2}{4}(|\psi_1|-|\psi_p|)^2.$$

From theorem 3.1

$$p2\Phi - \left[E\left(\frac{SSD}{diam}(\overline{G})\right)\right]^2 \leq \frac{p^2}{4}(|\psi_1|-|\psi_n|)^2,$$

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \geq \sqrt{2p\Phi - \frac{p^2}{4}(|\psi_1|-|\psi_p|)^2}.$$

Theorem 3.9. Let r_i *and* $s_i, 1 \leq i \leq p$ be positive real numbers, then by [13]

$$|p\sum_{i=1}^{p} r_i s_i - \sum_{i=1}^{p} r_i \sum_{i=1}^{p} s_i| \leq \mu(p)(A-a)(B-b)$$

where a, b, A and B are real constants, that for each $i, 1 \leq i \leq p, a \leq ai \leq A$ *and* $b \leq bi \leq B$. Further,

$$\mu(p) = p\lfloor \tfrac{p}{2} \rfloor \left(1 - \tfrac{1}{p}\lfloor \tfrac{p}{2} \rfloor\right).$$

Theorem 3.10. For a graph G with p vertices, we have

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \geq \sqrt{2p\Phi - \mu(p)\left(|\psi_1| - |\psi_p|\right)^2}.$$

Proof. Consider a graph G with p vertices $let\ |\psi 1|\ and\ |\psi p|$ are the maximum and minimum eigen values among $all\ |\psi_i|'s$ of $\frac{SSD}{diam}(\bar{G})$ respectively.
From theorem 3.9,

$$|p\sum_{i=1}^{p} r_i s_i - \sum_{i=1}^{p} r_i \sum_{i=1}^{p} s_i| \leq \mu(p)(A-a)(B-b).$$

Let $r_i = s_i = |\psi_i|, A = B = |\psi_1|, a = b = |\psi_n|$ then

$$|p\sum_{i=1}^{p}|\psi_i|^2 - \left[\sum_{i=1}^{p}|\psi_i|\right]^2 | \leq \mu(p)\left(|\psi_1| - |\psi_p|\right)\left(|\psi_1| - |\psi_p|\right).$$

From theorem 3.1

$$|p2\Phi - \left[E\left(\frac{SSD}{diam}(\overline{G})\right)\right]^2 | \leq \mu(p)\left(|\psi_1| - |\psi_p|\right)^2,$$

$$E\left[\frac{SSD}{diam}(\overline{G})\right] \geq \sqrt{2p\Phi - \mu(p)\left(|\psi_1| - |\psi_p|\right)^2}.$$

4. Square sum degree divided by diameter ma-trix of some complement graphs and its en-ergy.

Theorem 4.1. Let Cp, p ≥ 4 be the complement of cycle ,then

$$E\left[\frac{SSD}{diam}(\overline{C}_p)\right] = p^3 + 4p^2 - 77p + 192.$$

Proof. The complement of cycle Cp has its sum degree divided by diameter matrix as follows,

$$\frac{SSD}{diam}(\overline{C}_p) = \begin{bmatrix} 0 & (p-3)^2 & (p-3)^2 & \dots & (p-3)^2 \\ (p-3)^2 & 0 & (p-3)^2 & \dots & (p-3)^2 \\ (p-3)^2 & (p-3)^2 & 0 & \dots & (p-3)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (p-3)^2 & (p-3)^2 & (p-3)^2 & \dots & 0 \end{bmatrix}.$$

□

The characteristic polynomial is,

$$[\psi-(11p^2-92p+201)]\,[\psi+(p^2-6p+9)]^{(p-1)}=0.$$

$$\frac{SSD}{diam}(\overline{C}_p)=\begin{pmatrix} 11p^2-92p+201 & -(p^2-6p+9) \\ 1 & p-1 \end{pmatrix}.$$

Therefore

$$E\left[\frac{SSD}{diam}(\overline{C}_p)\right] = \left| 11p^2-92p+201 \right| 1 + \left| -(p^2-6p+9) \right| (p-1)$$
$$= p^3+4p^2-77p+192.$$

Theorem 4.2. Let Sp, p ≥ 3 be the complement of crown graph with 2p ver-tices, then

$$E\left[\frac{SSD}{diam}(\overline{S}_p^0)\right]=2p^3+22p^2-94p+120.$$

Proof. The complement of crown graph Sp with p vertices has its sum degree divided by diameter matrix as follows,

$$\frac{SSD}{diam}(\overline{S}_p^0)=\begin{bmatrix} 0 & p^2 & p^2 & \dots & p^2 \\ p^2 & 0 & p^2 & \dots & p^2 \\ p^2 & p^2 & 0 & \dots & p^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p^2 & p^2 & p^2 & \dots & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$$[\psi-(23p^2-94p+120)][\psi-(-p^2)]^{2p-1}=0,$$

$$\frac{SSD}{diam}(\overline{S}_p^0)=\begin{pmatrix} 23p^2-94p+120 & -p^2 \\ 1 & 2p-1 \end{pmatrix}.$$

Therefore

$$E(\frac{SSD}{diam}(\overline{S}_p^0)) = \left| (23p^2-94p+120) \right| 1 + \left| (-p^2) \right| 2p-1$$

$$=2p^3+22p^2-94p+120.$$

Theorem 4.3. Let Kp,p be the complement of double star graph with p vertices, *then* $E\left[\frac{SSD}{diam}\left(\overline{K}_{p,p}\right)\right] =$

$$\left(\frac{1}{3}\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396}\right)+\left(\frac{1}{3}(16p^3-54p^2+60p-22)\right)$$

Proof. The complement of double star graph $\overline{K}_{p,p}$ with p-vertices has its square sum degree divided by diameter matrix as follows

$$\frac{SSD}{diam}(\overline{K}_{p,p}) = \begin{bmatrix} 0 & \frac{5(p-1)^2}{3} & \frac{5(p-1)^2}{3} & \frac{5(p-1)^2}{3} & \frac{2(p-1)^2}{3} & \frac{5(p-1)^2}{3} & \dots & \frac{5(p-1)^2}{3} \\ \frac{5(p-1)^2}{3} & 0 & \frac{8(p-1)}{3} & \frac{8(p-1)}{3} & \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \dots & \frac{8(p-1)}{3} \\ \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & 0 & \frac{8(p-1)}{3} & \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \dots & \frac{8(p-1)}{3} \\ \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \frac{8(p-1)}{3} & 0 & \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \dots & \frac{8(p-1)}{3} \\ \frac{2(p-1)^2}{3} & \frac{5(p-1)^2}{3} & \frac{5(p-1)^2}{3} & \frac{5(p-1)^2}{3} & 0 & \frac{5(p-1)^2}{3} & \dots & \frac{5(p-1)^2}{3} \\ \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \frac{8(p-1)}{3} & \frac{8(p-1)}{3} & \frac{5(p-1)^2}{3} & 0 & \dots & \frac{8(p-1)}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \frac{8(p-1)}{3} & \frac{8(p-1)}{3} & \frac{5(p-1)^2}{3} & \frac{8(p-1)}{3} & \dots & 0 \end{bmatrix}$$

Its characteristic polynomial is

$$\left[\psi-\left(\tfrac{1}{3}(138p^2-692p+938+\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396})\right)\right]$$
$$\left[\psi-\left(\tfrac{1}{3}(138p^2-692p+938-\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396})\right)\right]$$
$$\left[\psi+\frac{2p^2-4p+2}{3}\right]\left[\psi+\frac{8p^2-16p+8}{3}\right]^{2p-3}=0$$

$$\frac{SSD}{diam}(\overline{K}_{p,p}) = \begin{pmatrix} \frac{1}{3}(138p^2-692p+938+\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396} \\ \frac{1}{3}(138p^2-692p+938-\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396} \\ \frac{2p^2-4p+2}{3} \\ \frac{8p^2-16p+8}{3} \end{pmatrix}$$

Therefore $E(\frac{SSD}{diam}(\overline{K}_{p,p})) =$

$$\left|\tfrac{1}{3}(138p^2-692p+938+\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396}\right|$$
$$1+\left|\tfrac{1}{3}(138p^2-692p+938-\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396}\right|$$
$$1+\left|\frac{2p^2-4p+2}{3}\right|1+\left|\frac{8p^2-16p+8}{3}\right|(2p-3)$$
$$=\left(\frac{1}{3}\sqrt{18044p^4-190992p^3+818792p^2-1903920p+1677396}\right)+\left(\frac{1}{3}(16p^3-54p^2+60p-22)\right)$$

REFERENCES

1. N. Biggs, Algebraic Graph Theory, 2nd Edition, Cambridge Univ. Press,Cambridge, 1993.
2. I.Gutman, The energy of graph. Ber. Math. Stat. Sekt. Forschungsz. Graz 103: 1-22, 1978.
3. I. Gutman, O. E. Polansky, Mathematical concepts in organic chemistry, Springer - verlag, Berlin, 1986.
4. I. Gutman, The energy of a graph: old and new results. In: Betten A., Kohnert A., Laue R., Wassermann A., editors. Algebraic combinatorics and applications, Berlin: Springer-Verlag, pp. 196-211, 2001.
5. A. Graovac, I. Gutman and N. Trinajstic, Topological approach to the chemistry of conjugated molecules, Berlin: Springer-Verlag, 1977.

6. I. Gutman, ”Topology and stability of conjugated hydrocarbons. The de-pendence of total Π- electron energy on molecular topology”, Journal of the Serbian Chemical Society, vol. 70, no. 3, pp. 441-456, 2005.
7. I. Gutman, “Topological studies on hetero conjugated molecules,” Z. Naturforch, vol. 45, pp. 1085-1089, 1990.
8. I. Gutman and M. Mateljevic, “Note on the Coulson integral formula,” Journal of Mathematical Chemistry, vol. 39, no 2, pp. 259-266, 2006.
9. X. Li, Y. Shi and I. Gutman, Graph Energy, New York: Springer, 2010.
10. Y. Liu, L. Zuo and C. Shang, “Chemical indices of generalized Peterson graph,” IAENG International Journal of Applied Mathematics, vol. 47, no. 1, pp. 123-129, 2017.
11. Y. Mi, L. Zuo and C. Shang, “On minimal energy of a class of bicyclic graph with given cycles length and pendant vertices,” IAENG International Journal of Applied Mathematics, vol. 47, no. 1, pp. 97105, 2017.
12. S. Pirzada and I. Gutman, “Energy of a graph is never the square root of an odd integer,” Applicable Analysis and Discrete Mathematics, vol. 2, no 1, pp. 118-121, 2008.
13. B. Zhou, “The energy of a graph,” MATCH Communications in Mathe-matical and in Computer Chemistry, vol. 51, pp. 111-118, 2004.
14. Fatemeh Alinaghipour a , Bahman Ahmadi ,On the energy of complement of regular line graphs ,Communications in Mathematical and in Computer Chemistry-60 (2008) 427-434
15. Manasa H S and Kavita Permi, Pradeep Kumar R “Square Sum Degree Divided By Diameter Energy of Graph” Liberte Journal, ISSN: 0024-2020, Vol. 8 Issue. 1 , 2020