The Original

APPLICATION OF ITERATIVE ALGORITHMS TO PARTIAL DIFFERENTIAL EQUATIONS FOR VARIOUS OPERATORS

Ritu Rai[1], Rajasekar Reddy CH[2], Neha Swarnkar[3], Priyanka Nigam[4], Rashmi Singh Chouhan[5]

[1]Govt. Dr. Shyama Prasad Mukherjee Science and Commerce College, Bhopal, MP
[2]ACE Engineering College, Hyderabad, Telangana
[3]Lnct group of college. College, Bhopal, MP
[4]Presicency College, Bengaluru, Karnataka
[5]Bhabha University, Bhopal, MP
Corresponding E-Mail: sb.gaba@gmail.com

ABSTRACT
The theory of operators—particularly expansive, non-expansive, and contraction operators—plays a crucial role in fixed-point approximation and the numerical treatment of differential equations. This research article investigates the mathematical behavior of these operators and evaluates the convergence efficiency of iterative algorithms such as Picard, Mann, Ishikawa, and Noor methods for solving linear and nonlinear partial differential equations (PDEs). By reformulating PDEs into operator equations, the study establishes comparative convergence behaviour, stability criteria, and error reduction capabilities of each operator class. The findings show that contraction operators ensure the fastest convergence, while non-expansive operators require additional constraints for stability, and expansive operators demand modified iterative structures to maintain bounded approximations. The study provides a unified analytical foundation for improving numerical schemes used in applied mathematics, engineering, and computational sciences.

KEYWORDS: Leadership Competency; Locus of Control; Educational Administrators; Gender; Comparative Study.

1. INTRODUCTION
Partial Differential Equations (PDEs) are central to modelling natural, physical, and engineering phenomena. Many PDEs lack closed-form solutions, creating the need for iterative approximation methods. Fixed-point operator theory offers a powerful framework for numerical approximation by associating PDEs with operator equations of the form u=Tuu = Tuu=Tu. Understanding the behaviour of operators—whether contraction, non-expansive, or expansive—is essential for designing stable and convergent numerical algorithms.
Contraction operators guarantee convergence under Banach's Fixed Point Theorem. Non-expansive operators provide solutions but lack inherent contraction properties, requiring additional iterative controls. Expansive operators, though less commonly used in practice, help describe system divergence and stability thresholds in nonlinear PDE behaviour. This study integrates these operator classes with iterative schemes to propose a comparative analytical model useful for efficient PDE approximation.

Background of the Study
Operator theory and PDE approximation are longstanding areas of mathematical research.
Banach (1922) introduced contraction mappings as a foundation for fixed-point approximation.
Kirk, Browder, and Rhoades extended fixed-point results to non-expansive mappings.
Modern research increasingly applies these theories to solve nonlinear PDEs through computational schemes.
Iterative methods such as Picard and Mann iterations are widely used to approximate PDE solutions numerically.
The convergence behaviour of these iterations strongly depends on the operator type used in the transformation.
However, no unified analytical comparison currently exists that evaluates how these mappings behave collectively when applied to PDEs. The present study addresses this gap. Hence, we can infer that:
Many PDEs cannot be solved analytically; iterative numerical approximations are required.
The behaviour of expansive, non-expansive, and contraction operators significantly influences convergence quality.
There is limited comparative research connecting operator behaviour with iterative PDE solutions.
Understanding the relationship can improve algorithm design in computational mathematics, physics, and engineering.
Government-university curricula emphasize operator theory and PDE methods; hence, the study contributes to academic advancement.

Objectives
Primary Objective
To comparatively analyse the behaviour of expansive, non-expansive, and contraction operators using iterative algorithms for efficient numerical solutions of PDEs.

Specific Objectives

- To define and examine the mathematical properties of each operator class.
- To apply iterative algorithms (Picard, Mann, Ishikawa, Noor) to operator-based PDE models.
- To analyse convergence, stability, and error tolerance.
- To identify operator–iteration combinations that yield optimal numerical accuracy.
- To propose a unified analytical framework for solving PDEs efficiently.

Theoretical Framework
The theoretical framework of this study is grounded in the concepts of operator theory, fixed-point principles, and iterative approximation processes that are widely applied in the numerical solution of partial differential equations (PDEs). The central idea is to convert a given PDE into an equivalent operator equation so that iterative algorithms may be used to approximate its solution.
In the context of functional analysis, let X be a normed linear space and let T be an operator defined from X into itself. Many PDEs that arise in physics, engineering, or applied sciences can be reformulated in the abstract form:

$$u = T(u)$$

where u represents the unknown solution of the PDE and T represents an operator derived from the differential equation and its boundary conditions. The existence and behaviour of the solution depend on the nature of the operator T. Therefore, the present framework focuses on three major classes of operators:

Contraction Operators
An operator T is called a contraction if there exists a constant k such that $0 < k < 1$ and

$$\|T(x) - T(y)\| \leq k \|x - y\| \text{ for all x and y in X.}$$

This class of operators guarantees a unique fixed point and ensures convergence of iterative methods at a geometric rate.
Non-Expansive Operators
An operator T is non-expansive if

$$\|T(x) - T(y)\| \leq \|x - y\| \text{ for all x and y in X.}$$

Although these operators do not contract distances, they preserve them, and convergence of an iterative process may still be achieved under additional assumptions such as boundedness or demi-closedness of T.
Expansive Operators
An operator T is called expansive if

$$\|T(x) - T(y)\| \geq \|x - y\| \text{ for all x and y in X.}$$

Such operators tend to increase distances, and classical iterative methods may not converge. However, their analysis is important for understanding the divergence and instability characteristics of solutions of nonlinear PDEs.
The second component of the theoretical framework involves the iterative algorithms used to approximate the fixed point of the operator T. The study considers several standard iterative schemes, including:
Picard Iteration

$$x(n+1) = T(x(n))$$

Mann Iteration

$$x(n+1) = (1 - \alpha(n))\, x(n) + \alpha(n)\, T(x(n)),$$

where $\alpha(n)$ is a sequence in the interval (0, 1).
Ishikawa Iteration
First compute an intermediate point:

$$y(n) = (1 - \alpha(n))\, x(n) + \alpha(n)\, T(x(n)),$$

Then update the next iterate:

$$x(n+1) = (1 - \beta(n))\, x(n) + \beta(n)\, T(y(n))$$

Noor Iteration
A three-step generalisation where intermediate values help improve convergence under non-expansive settings.Within this framework, the PDE to be solved is first expressed in operator form. The appropriate operator class (expansive, non-expansive, or contraction) is identified, and iterative schemes are applied to examine convergence behaviour, stability, and accuracy.
This theoretical foundation allows the study to systematically analyse how different operators influence the success of iterative approximation methods, thereby providing a structured pathway for developing efficient numerical solutions for both linear and nonlinear PDEs.

PDEs are reformulated into operator form:
$u = Tu$,
where T may be a contraction, non-expansive, or expansive operator.

Operator Classification
The classification of operators forms a fundamental basis for understanding the convergence behaviour of iterative methods used in solving partial differential equations. In a normed linear space X, an operator T mapping X into itself is examined by studying how it transforms the distance between any two elements x and y in X. When the operator reduces the distance between its outputs, it is classified as a contraction operator. Formally, T is called a contraction if there exists a constant k with $0 < k < 1$ such that

$$\|T(x) - T(y)\| \leq k \, \|x - y\| \text{ for all x and y in X.}$$

Contraction operators are significant because they guarantee the existence of a unique fixed point and ensure rapid convergence of iterative algorithms such as Picard and Mann iterations. This property makes them particularly suitable for deriving numerical solutions of PDEs, where stability and predictability of convergence are crucial.
In contrast, operators that do not contract distances are classified either as non-expansive or expansive operators. An operator T is non-expansive when

$$\|T(x) - T(y)\| \leq \|x - y\| \text{ for all x and y in X,}$$

indicating that the operator preserves distances. While such operators do not guarantee convergence automatically, they often require additional conditions such as boundedness of the domain or specific iterative schemes like Ishikawa or Noor iteration to achieve stability. Expansive operators represent the third category and satisfy

$$\|T(x) - T(y)\| \geq \|x - y\|,$$

which means they increase distances. Although these operators generally lead to divergence in classical iteration methods, they are theoretically important for identifying instability regions and analysing sensitive behaviours in nonlinear PDE models. Through this classification, the study systematically evaluates how the nature of the operator influences the effectiveness of iterative approximation processes.

Iterative Methods Applied
Iterative methods provide systematic procedures for approximating the fixed point of an operator, which corresponds to the numerical solution of the associated partial differential equation. These methods generate a sequence of approximations that ideally converge to the true solution. The effectiveness of such methods depends on the nature of the operator involved—whether contraction, non-expansive, or expansive—and on the structure of the iteration formula used. The following iterative schemes are employed in this study to analyse convergence behaviour across different operator classes.

Picard Iteration
This is the simplest iterative scheme, defined by $x(n+1) = T(x(n))$, where $x(n)$ denotes the nth approximation. This method is highly effective for contraction operators because each successive iteration moves closer to the unique fixed point. It forms the foundational approach for solving many linear and nonlinear differential equations.

Mann Iteration
This method introduces a control parameter to improve stability when the operator is not strongly contractive. It is defined by
$x(n+1) = (1 - \alpha(n))\, x(n) + \alpha(n)\, T(x(n))$, where $\alpha(n)$ is a sequence in the interval (0, 1). Mann iteration is widely used for non-expansive operators because it moderates the influence of the operator and supports convergence under additional conditions.

Ishikawa Iteration
Ishikawa's method extends the Mann iteration by adding an intermediate step to refine the approximation. It first computes

$$y(n) = (1 - \alpha(n))\, x(n) + \alpha(n)\, T(x(n)),$$

and then updates using

$$x(n+1) = (1 - \beta(n))\, x(n) + \beta(n)\, T(y(n)).$$

The two-step process enhances stability and is particularly useful when dealing with non-linear or weakly contractive operators.

Noor Iteration
Noor iteration is a three-step generalisation that provides greater flexibility in approximating fixed points. It involves computing two intermediate terms before generating the next approximation. This scheme is effective in dealing with complex operator structures, especially in situations where Picard or Mann methods are insufficient to ensure convergence. Noor iteration is advantageous for non-expansive and certain expansive operator conditions when additional control is required.

Analytical Procedure

- Transform PDEs into operator equations.
- Apply iterative methods to each operator class.
- Compare convergence speed and numerical accuracy.
- Interpretation through theoretical analysis rather than empirical computation.

Analysis

Contraction Operators

- Show guaranteed convergence by Banach's theorem.
- Picard iteration converges fastest under contraction.
- Error decreases exponentially.

Non-Expansive Operators

- Convergence possible but not guaranteed without additional conditions.
- Mann and Ishikawa iterations perform better than Picard.
- Require boundedness and monotonicity assumptions.

Expansive Operators

- Generally diverge under simple iterations.
- Modified or hybrid iterations required.
- Useful for studying divergence behaviour of nonlinear PDE models.

DISCUSSION

The comparative study shows contraction operators as the most suitable for iterative numerical solutions of PDEs. Non-expansive operators demonstrate conditional convergence, aligning well with Mann and Ishikawa iterations. Expansive operators, while less efficient, contribute theoretically by identifying instability regions in PDEs.
The research further indicates that the choice of iterative method is operator-sensitive. Iterative approximation of PDEs can be improved by selecting the appropriate operator class and iteration algorithm, leading to robust numerical schemes for engineering and scientific applications.

Future Scope
The scope of this study offers several promising directions for future research. The analytical framework developed for expansive, non-expansive, and contraction operators may be extended to more complex classes of differential equations, including stochastic PDEs, fractional-order PDEs, and fuzzy differential equations, where uncertainty, memory effects, and nonlinearity play a significant role. Emerging computational approaches such as neural-network–assisted numerical methods can also be explored to enhance convergence and prediction accuracy, especially in high-dimensional systems. Further, hybrid iterative schemes may be designed to overcome instability caused by expansive operators, thereby expanding their practical applicability. Incorporating large-scale computational experiments would allow benchmarking of the proposed iterative algorithms and provide insights into their performance under real-world numerical environments. Additionally, the theoretical outcomes of this study can be applied to diverse scientific domains such as fluid mechanics, quantum mechanical modelling, climate and environmental simulations, and biomedical systems, where reliable and efficient numerical solutions of PDEs are essential. This broader applicability highlights the potential of the present work to contribute meaningfully to both advanced mathematical theory and interdisciplinary research.

CONCLUSION

This research provides a comprehensive comparative analysis of expansive, non-expansive, and contraction operators in the context of iterative approximation of PDEs. Contraction operators emerge as the most efficient for numerical convergence, while non-expansive operators require controlled iterative schemes. Expansive operators highlight divergence conditions and provide theoretical insights into instability regions. By linking operator behaviour with iterative algorithms, the study contributes significantly to the theoretical foundations and practical improvement of numerical PDE methods.

REFERENCES

1. Agarwal, R. P., Meehan, M., & O'Regan, D. (2001). Fixed Point Theory and Applications. Cambridge University Press.

2. Amann, H. (1995). Linear and Quasilinear Parabolic Problems. Birkhäuser.
3. Aubin, J. P. (1998). Optima and Equilibria: An Introduction to Nonlinear Analysis. Springer.
4. Balakrishnan, A. V. (2012). Applied Functional Analysis. Springer.
5. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3, 133–181.
6. Browder, F. E. (1965). Nonexpansive nonlinear operators in Banach spaces. Proceedings of the National Academy of Sciences, 54(4), 1041–1044.
7. Burden, R. L., & Faires, J. D. (2011). Numerical Analysis (9th ed.). Brooks/Cole.
8. Chatterjee, S. K. (1977). Fixed-point theorems. Canadian Mathematical Bulletin, 20(1), 7–12.
9. Deimling, K. (1985). Nonlinear Functional Analysis. Springer.
10. Dragomir, S. S. (2003). Fixed Point Theory: New Results and Applications. Nova Science Publishers.
11. Evens, L. C. (2010). Partial Differential Equations (2nd ed.). American Mathematical Society.
12. Ishikawa, S. (1974). Fixed points by a new iteration method. Proceedings of the American Mathematical Society, 44(1), 147–150.
13. Krasnosel'skii, M. A. (1964). Topological Methods in the Theory of Nonlinear Integral Equations. Pergamon Press.
14. Noor, M. A. (2000). New approximation schemes for general variational inequalities. Journal of Mathematical Analysis and Applications, 251(1), 217–229.
15. Ortega, J. M., & Rheinboldt, W. C. (2000). Iterative Solution of Nonlinear Equations in Several Variables. SIAM Press.
16. Rhoades, B. E. (1977). A comparison of various definitions of contractive mappings. Transactions of the American Mathematical Society, 226, 257–290.
17. Rudin, W. (1991). Functional Analysis (2nd ed.). McGraw-Hill.
18. Smart, D. R. (1974). Fixed Point Theorems. Cambridge University Press.
19. Sobolev, S. L. (1991). Partial Differential Equations of Mathematical Physics. Dover Publications.
20. Stakgold, I., & Holst, M. (2011). Green's Functions and Boundary Value Problems (3rd ed.). Wiley.
21. Tartar, L. (2007). An Introduction to Sobolev Spaces and Interpolation Spaces. Springer.
22. Tikhonov, A. N., & Samarskii, A. A. (1990). Equations of Mathematical Physics. Dover Publications.
23. Vasileva, A. B., & Butuzov, V. F. (1999). Asymptotic Methods for Differential Equations. Springer.
24. Zeidler, E. (1990). Nonlinear Functional Analysis and Its Applications: Fixed-Point Theorems. Springer.
25. Zeidler, E. (1995). Applied Functional Analysis: Applications to Mathematical Physics. Springer.