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GENERALIZED FRÉCHET MOMENT EXPONENTIAL DISTRIBUTION: MATHEMATICAL PROPERTIES, ESTIMATION, AND APPLICATIONS

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ABSTRACT

In this study, we introduced a new extension of the moment exponential distribution using the Fréchet family of distributions. The new model is named the “generalized Fréchet moment exponential (GFME) distribution”. We derived its mathematical properties, including moments, incomplete moments, probability weighted moments, moment generating function, entropies, quantile function, mean residual life, mean inactivity time, stress strength reliability, and order statistics. Moreover, the parameter estimation of the GFME distribution is discussed using the renowned maximum likelihood estimation and four distance-based approaches. A detailed Monte Carlo simulation study is utilized to illustrate the estimation behavior of the derived estimators using different sample sizes and choices of parameters. It is found that the bias and mean squared error are decreased with increasing sample sizes. The applicability and flexibility of the GFME distribution are illustrated using four asymmetric datasets from different fields. The proposed GFME model efficiently analyzed these datasets as compared to the considered renowned models, including other generalizations of the moment exponential distribution.

KEYWORDS: GFME distribution, moments, entropies, Reliability analysis, estimation

1. INTRODUCTION

Probability distributions are used in central processing tasks of statistical modeling and inference, and risk quantification. Classical distributions typically do not reflect complicated aspects of real-world data, especially skewness, heavy tails, non-monotonic hazard rate patterns, and so on, in many application fields, including hydrology, finance, insurance, reliability engineering, and environmental sciences. To overcome these shortcomings, an extensive literature has been devoted to the need to establish a new family of probability distributions by enriching existing well-known baseline models by adding more shape parameters.

The generator approach, in which a cumulative distribution function (CDF) served as the baseline, is embedded into a more general parametric framework to provide flexibility and is one such power tool in this direction. The most notable addition is the exponentiated-G family by Gupta et al. (1998), where the baseline CDF is elevated to a positive power, producing more shaped shapes. It has since been followed by various significant generator mechanisms such as the Marshall-Olkin family (Marshall and Olkin, 1997), the Beta-G family (Eugene et al., 2002), the Gamma-generated family (Zografos and Balakrishnan, 2009), and the Kumaraswamy-G family (Cordeiro and de Castro, 2011). More extensions like the Exponentiated generalized-G family, Weibull-G family, and various transmuted and odd-logistic constructions have extended the portfolio of modeling tools that statisticians now possess. The Fréchet distribution is a classical extreme value distribution, taking a leading role in the context of extreme value theory. It is mostly appropriate when it comes to the modeling of maxima and heavy-tailed phenomena. It is very applicable in risk analysis and actuarial science because its ability to describe upper tail behavior is very flexible. Having this in mind, Haq and Elgarhy (2018) suggested the odd Fréchet-G family (OFr-G), which is a generator built by the modeling of the odds ratio $G(x;\xi)/(1-G(x;\xi)$ into the Fréchet distribution. The resulting construction is a new class of distributions whose density and hazard rate functions can all have left-skewed, symmetric, reversed J-shaped, and monotone behavior based on the choices of parameters. Notably, the OFr-G system maintains the aspect of tractability, but the tail and shape features are greatly improved. At this point, even though the OFr-G family offers a flexible framework, it would be desirable to do further refinements to support data sets with strong skew, heavy tails, or rich moment structure that cannot be well represented by existing sub-models, including Odd Fréchet-Weibull, Odd Fréchet-Lomax, or Odd Fréchet-Gamma distributions. Specifically, by adding more moment-based flexibility to the Fréchet generated scheme, a model with better control on dispersion and higher order properties can be obtained.

In this paper, we introduce a new member of the Odd Fréchet-G family (OFr-G), namely the GFME distribution. The CDF and probability density function (PDF) of OFr-G generators are as follows:

$$F(x;\theta)=\exp\left[-\left(\frac{1-G(x)}{G(x)}\right)^{\theta}\right], \tag{1}$$

and

$$f(x;\theta)=\frac{\theta g(x)\big(1-G(x)\big)^{\theta-1}}{\big(G(x)\big)^{\theta+1}}\exp\left[-\left(\frac{1-G(x)}{G(x)}\right)^{\theta}\right],\qquad x,\theta>0. \tag{2}$$

The proposed model is constructed by applying the Odd Fréchet-G generating technique to the moment exponential baseline distribution. The CDF and PDF of the Moment Exponential (MEx) distribution are given by

$$G(x;\beta)=1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}},\quad x>0\ \&\ \beta>0, \tag{3}$$

and

$$g(x;\beta)=\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}. \tag{4}$$

The exponential-type structure can be incorporated into the odd Fréchet mechanism to give the new distribution the heavy tail and extreme value properties of the Fréchet generator, with the added advantage of the analytical simplicity and interpret-ability of the exponential base. This extra shape parameter of the odd Fréchet transformation greatly increases the flexibility of the modeling of skew, kurtosis, and dynamics of hazard rates. The key objectives of our study are.

- Introduce a new probability distribution capable of effectively modeling skewed datasets.
- Derive and investigate key statistical properties, including moments, reliability characteristics, order statistics, stochastic ordering, and record values.
- Develop parameter estimation using renowned maximum likelihood and four distance based methods such as Anderson Darling, Cramer von Mises, Ordinary least squares, and weighted least squares.
- Assess the goodness-of-fit of the GFME distribution through real data applications and compare it with existing distributions.

The remainder of the paper is organized as follows. Section 2 presents the derivation of the new distribution and its Reliability characteristics. Section 3 investigates several important Statistical properties of the distribution. Statistical inference procedures are developed in Section 4. Section 5 examines the flexibility and applicability of the proposed unit model using two skewed datasets. Finally, Section 6 provides concluding remarks along with possible directions for future research.

2. The GFME Distribution: Formulation and Reliability Properties

This section formally defines and studies a new member of the OFr-G class, which we call the Generalized Fréchet Moment Exponential distribution. The model is derived through the inclusion of the MEx baseline distribution into the OFr generating model. We obtain the main structural aspects of the distribution, such as explicit forms of its CDF, PDF, a linear representation of PDF, survival function, the hazard rate function, and the second rate of failure. The parameters of the graphical analysis are used to elaborate on these analytical results to illustrate the range of shapes that can be attained using different parameter settings.

The CDF of the GFME distribution is obtained by substituting equation (3) into equation (1), and is given by

$$F(x;\beta,\theta)=\exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]. \tag{5}$$

The PDF of the GFME distribution is obtained by substituting equations (3) and (4) into equation (2) and is expressed as follows.

$$f(x;\beta,\theta)=\frac{\theta\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\left[\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}}\exp\left\{-\left[\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right]^{\theta}\right\},\quad x>0\ \&\ \beta,\theta>0. \tag{6}$$

Equation (6) defines a two-parameter distribution in which β acts as a scale parameter inherited from the Moment Exponential baseline, while θ is the additional shape parameter introduced through the Odd Fréchet transformation. The interplay of these parameters determines the skewness, tail behavior, and the modality of the density function.

As shown in Fig. 1, the GFME distribution has a very diverse variety of shapes in the combination of β and θ. The density is unbiased (right skewed), decreasing in shape with strong tail behavior on small values of θ. With an increase in θ, the distribution approaches a clear unimodal distribution with higher concentration and steeper peaks around the mode. The aspect of scale and dispersion is mainly determined by parameter 8, which shifts the peak to the right and flattens the curve with an increase in the value of 8. In general, the graphical findings support the fact that the GFME distribution is very versatile, and it can characterize declining, skewed to the right, and unimodal density forms.

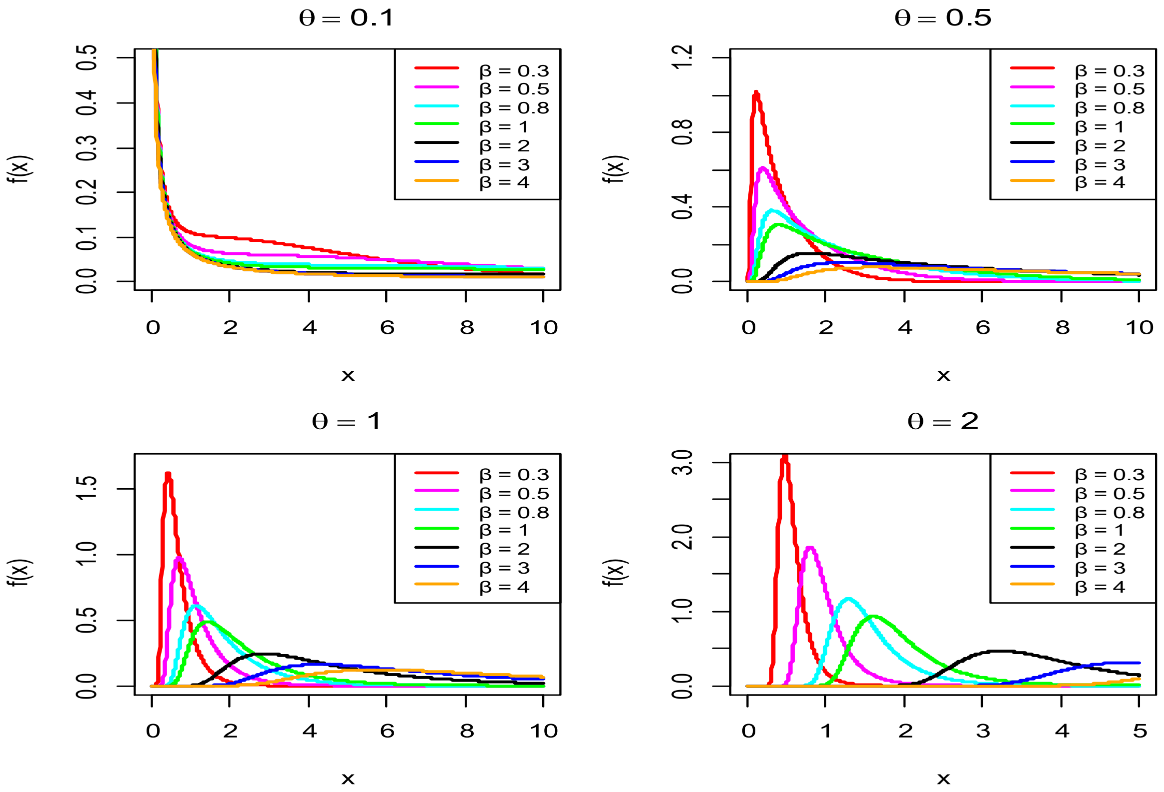

Figure 1: Shape Behavior of the GFME Density Function under different Parameter configurations

2.1 Linear representation

In this section, the linear representation of the PDF and CDF of the new distribution is discussed. Linear representation is useful when deriving the statistical properties of the model.

Using the Maclaurin series expansion to equation (6), the PDF in equation (6) can be written as follows

$$f(x;\beta,\theta)=\left[\frac{\theta\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\left(\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}}\right]\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta i}$$

Further simplifying

$$f(x;\beta,\theta)=\theta\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\frac{\left(\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta i+\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta i+\theta+1}}$$

Using binomial expansions twice yields

$$f(x;\beta,\theta)=\theta\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i}}{i!}\binom{j+\theta i+\theta}{\theta i+\theta}\binom{j+\theta i+\theta-1}{k}\left(\frac{x}{\beta}\right)^{k}\left(e^{-\frac{x}{\beta}}\right)^{\theta i+\theta-1+j}$$

For convenience, $f(x;\beta,\theta)$ can be written as

$$f(x;\beta,\theta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\,\frac{\theta x^{k+1}}{\beta^{k+2}}e^{-x\left(\frac{\theta i+\theta+j}{\beta}\right)} \tag{7}$$

The linear representations of PDF are obtained by using the following expansions.

$$e^{-x}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}x^{i},\qquad (1-z)^{-k}=\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}z^{j}\ \ \&\ \ (1+z)^{b-1}=\sum_{k=0}^{\infty}\binom{b-1}{k}z^{k}$$

where

$$\varphi_{ijk}=\frac{(-1)^{i}}{i!}\binom{j+\theta i+\theta}{\theta i+\theta}\binom{j+\theta i+\theta-1}{k}.$$

2.2. Survival function

The survival function of the proposed model is expressed as follows.

$$S(x;\beta,\theta)=1-Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right].$$

2.3. Hazard and Reversed Hazard Rate Functions

The Hazard rate (hrf) and Reversed Hazard Rate (rhrf) of the GFME distribution are, respectively, given by the following equations

$$hrf(x;\beta,\theta)=\frac{\theta\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\left[\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{\theta-1}\exp\left\{-\left[\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right]^{\theta}\right\}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}\left(1-exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right)}$$

and

$$rhrf(x;\beta,\theta)=\theta\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\frac{\left[\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}}$$

Figure 2 indicates that the hazard rate of the GFME distribution has a high level of flexibility with the values of the parameters. In the case of very small values of θ (e.g., 0.1), the hazard rate would begin at a high level and then decline sharply, meaning that the hazard rate is strongly decreasing. As the θ varies from 0.5 to 1, the Hazard rate is monotone increasing, that is, it rises steeply at small x, then tends to increase gradually, implying that the failure tendency increases with time. When theta is larger (e.g., θ=2), the hazard grows quickly during the initial stages and then passes to the plateau, with a strong upward trend with potential leveling tendencies. In all the panels, low values of β increase the level of hazard, whereas high values of β reduce and dampen the level of hazard, which proves that β is a scale parameter. In general, distribution can be used to model declining and rising shapes of hazard, thus enjoying high flexibility in lifetime data analysis.

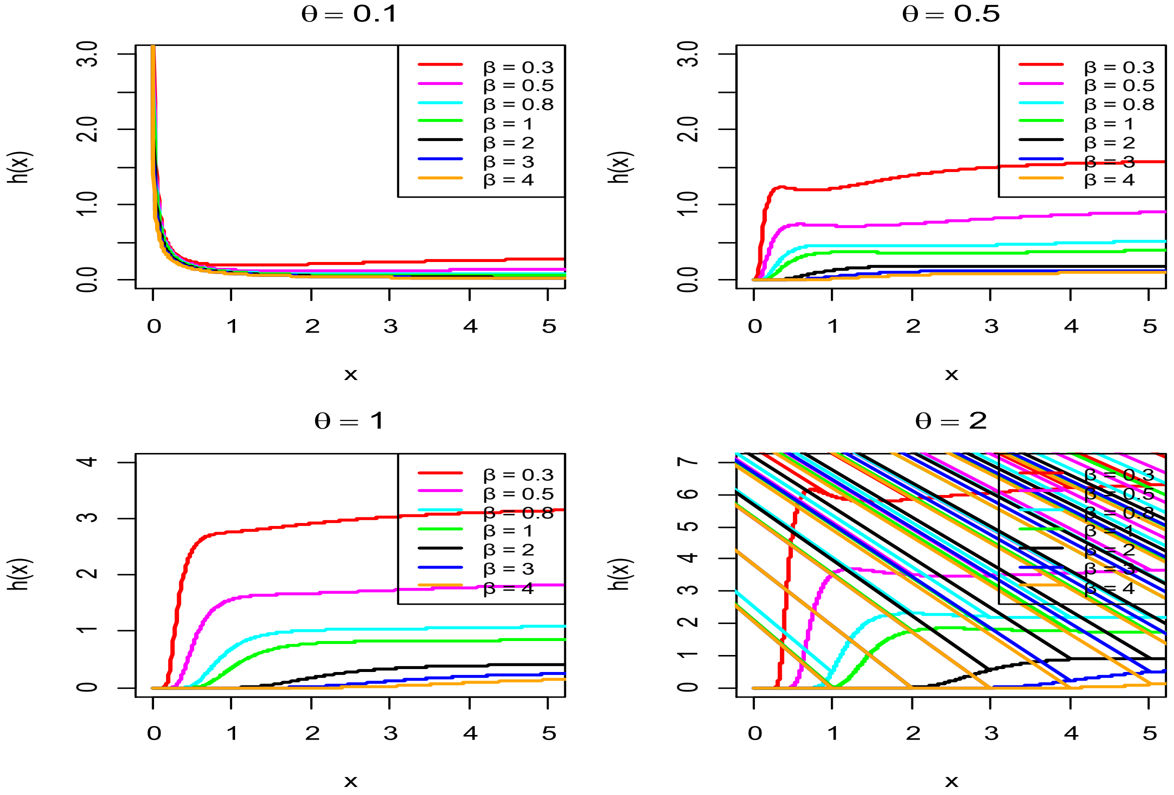

Figure 2: plots of the hazard rate of the GFME distribution for various parameter values

2.4. Odd function and Mills ratio

The odd function and Mills ratio

$$OR|x;\beta,\theta=\frac{F|x;\beta,\theta}{1-F|x;\beta,\theta}=\frac{1}{Exp\left[\left(\frac{(1+x/\beta)e^{-x/\beta}}{1-(1+x/\beta)e^{-x/\beta}}\right)^{\theta}\right]-1}.$$

and

$$MR|x;\beta,\theta = \frac{1-F|x;\beta,\theta}{F|x;\beta,\theta} == Exp\left[\left(\frac{(1+x/\beta)e^{-x/\beta}}{1-(1+x/\beta)e^{-x/\beta}}\right)^{\theta}\right]-1.$$

3. Statistical properties

This section derives and analyzes some significant statistical properties of the GFME distribution. These are the ordinary and central moments, and some similar descriptive measures like the mean, the variance, skewness, and kurtosis. We also get the moment generating function and others like generating functions where they exist. Moreover, the quantile function is computed to provide the random variation generation and analysis in percentiles. The uncertainty measures, such as the functions of entropy, are explored to determine the amount of information in the distribution. Lastly, the order statistics are determined, which are used in reliability analysis and inferential procedures applications.

3.1. Moments and related measures

Theorem 1: *If $X \sim GFME\ (x;\theta,\beta)$ with scale $(\beta > 0)$, shape $(\theta > 0)$ parameters, then the r-th moment about the origin (μ_r') of X is given by*

$$\mu_r' = \theta\beta^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(r+k+2)}{(\theta(i+1)+j)^{r+k+2}}$$

Proof: The r-th moment about the origin is defined as

$$\mu_r{}' = \int_0^{\infty} x^r f|x;\beta,\theta\, dx$$

By placing the information from (7) in μ_r' can be written as follows:

$$\mu_r' = \int_0^{\infty} x^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\ \frac{\theta x^{k+1}}{\beta^{k+2}}e^{-x\left(\frac{\theta i+\theta+j}{\beta}\right)}dx$$

$$\mu_r' = \frac{\theta}{\beta^{k+2}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\ \int_0^{\infty} x^{r+k+1}e^{-x\left(\frac{\theta i+\theta+j}{\beta}\right)}dx$$

Furthermore, we simplify the terms.

$$\mu_r' = \frac{\theta}{\beta^{k+2}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\ \int_0^{\infty}\left(\frac{y\beta}{\theta(i+1)+j}\right)^{r+k+1}\ e^{y}\frac{\beta}{\theta(i+1)+j}dy$$

and the simplified expression can be written as follows:

$$\mu_r' = \theta\beta^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(r+k+2)}{(\theta(i+1)+j)^{r+k+2}}$$

Hence proved.

The first four moments about the origin, denoted by $(\mu_1{}', \mu_2{}', \mu_3{}'\ and\ \mu_4{}')$, are obtained by substituting $r = 1,2,3,$ and 4, respectively, into the general expression for the r-th raw moment. These moments are provided as follows:

$$\mu_1{}' = \theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}$$

$$\mu_2{}' = \theta\beta^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}}$$

$$\mu_3{}' = \theta\beta^3 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+5)}{(\theta(i+1)+j)^{k+5}}$$

$$\mu_4{}' = \theta\beta^4 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+6)}{(\theta(i+1)+j)^{k+6}}$$

These expressions establish that the moments of the GFME distribution exist for all admissible parameter values and can be computed numerically through truncated summation.

The variance of the distribution is obtained from the standard moments relationship and provided as follows.

$$\sigma^2 = \theta\beta^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}} - \left(\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right)^2$$

Based on these moments, several important descriptive measures can be obtained. The dispersion index (DI), coefficient of variation (CV), coefficient of skewness (SK), and coefficient of kurtosis (CK) are respectively given by

$$DI = \frac{\theta\beta^2\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}} - \left(\theta\beta\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right)^2}{\theta\beta\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}}$$

$$CV=\frac{\sqrt{\theta\beta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}}-\left(\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right)^2}}{\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}}$$

$$\mathcal{SK}=\frac{\left[\begin{array}{l}\left[\theta\beta^3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+5)}{(\theta(i+1)+j)^{k+5}}\right]-3\left[\theta\beta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}}\right]\times\\ \left[\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right]+2\left[\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right]^3\end{array}\right]}{\left[\theta\beta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}}-\left(\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right)^2\right]^{\frac{3}{2}}}$$

and

$$\mathcal{CK}=\frac{\left[\begin{array}{l}\left[\theta\beta^4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+6)}{(\theta(i+1)+j)^{k+6}}\right]-4\left[\theta\beta^3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+5)}{(\theta(i+1)+j)^{k+5}}\right]\times\\ \left[\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right]+6\left[\theta\beta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}}\right]\times\\ \left[\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right]^2-3\left[\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right]^4\end{array}\right]}{\left[\theta\beta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+4)}{(\theta(i+1)+j)^{k+4}}-\left(\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_{ijk}\Gamma(k+3)}{(\theta(i+1)+j)^{k+3}}\right)^2\right]^2}$$

Table 1 shows the numerical values of various measures of descriptive statistics of the GFME distribution of chosen values of the parameters θ *and* β. These are the first four raw moments $({\mu_1}', {\mu_2}', {\mu_3}'$ *and* ${\mu_4}')$, and variance V(X), the dispersion index DI(X), the coefficient of skewness Sk(X) and the coefficient of kurtosis CK(X).

Based on the table, it is possible to note that the higher the scale parameter β is, the higher the values of the moments and their variance, which means that the distribution is spread and more varied. The dispersion index also rises with greater values of β, which indicates greater dispersion as compared to the mean. In addition, the skewness scores are positive in all the combinations of the parameters, which means that the skewness of the GFME is right-skewed. The values of the kurtosis are more than three, and this indicates that the distribution has a leptokurtic behavior, which means that the distribution has heavier tails than the normal distribution. These findings demonstrate that the GFME distribution is flexible to use in modeling positively skewed data with different levels of dispersion and tail behavio

Table 1: Some descriptive measures of GFME distribution

Parameters		**Measures**							
θ	β	μ_1'	μ_2'	μ_3'	μ_4'	$V(X)$	$DI(X)$	Sk(X)	$CK(X)$
0.5	0.5	1.0370	1.2394	1.7386	2.8931	0.1640	0.4050	1.7031	7.7626
	1.0	2.0740	4.9575	13.9088	46.2888	0.6562	0.8101	1.7031	7.7626
	1.5	3.1109	11.1544	46.9422	234.3371	1.4764	1.2151	1.7031	7.7626
	2.0	4.1479	19.8300	111.2703	740.6210	2.6248	1.6201	1.7031	7.7626
	2.5	5.1849	30.9844	217.3248	1808.1567	4.1012	2.0251	1.7031	7.7626
1.0	0.5	1.1561	1.7264	3.3138	7.9774	0.3897	0.6243	1.713	7.4917
	1.0	2.3123	6.9055	26.5107	127.6385	1.5589	1.2486	1.713	7.4917
	1.5	3.4684	15.5374	89.4736	646.1701	3.5075	1.8728	1.713	7.4917
	2.0	4.6245	27.6220	212.0855	2042.2165	6.2356	2.4971	1.713	7.4917
	2.5	5.7807	43.1593	414.2294	4985.8802	9.7431	3.1214	1.713	7.4917
1.5	0.5	1.0370	1.2394	1.7386	2.8931	0.1640	0.4050	1.7031	7.7626
	1.0	2.0740	4.9575	13.9088	46.2888	0.6562	0.8101	1.7031	7.7626
	1.5	3.1109	11.1544	46.9422	234.3371	1.4764	1.2151	1.7031	7.7626
	2.0	4.1479	19.8300	111.2703	740.6210	2.6248	1.6201	1.7031	7.7626
	2.5	5.1849	30.9844	217.3248	1808.1567	4.1012	2.0251	1.7031	7.7626
2.5	0.5	0.9491	0.9551	1.0299	1.2050	0.0542	0.2328	1.628	7.7123
	1.0	1.8983	3.8202	8.2389	19.2803	0.2167	0.4655	1.628	7.7123
	1.5	2.8474	8.5955	27.8064	97.6067	0.4876	0.6983	1.628	7.7123
	2.0	3.7966	15.2809	65.9114	308.4853	0.8669	0.9311	1.628	7.7123
	2.5	4.7457	23.8764	128.7333	753.1380	1.3545	1.1638	1.628	7.7123

3.1.1. Incomplete Moments

Theorem *2: If* $X \sim GFME\ (x;\theta,\beta)$ *with scale* $(\beta > 0)$, *shape* $(\theta > 0)$ *parameters, then the r-th incomplete moment about the origin* $(m_r(t))$ *of X is given by*

$$m_r(t) = \frac{\theta\beta^r}{(\theta(i+1)+j)^{r+k+2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \ .\gamma\left(\mathrm{r}+\mathrm{k}+2, \frac{t}{\beta}\theta(i+1)+j\right)$$

Proof: The r-th incomplete moment of a continuous random variable is defined as

$$m_r(t) = \int_0^t x^r f|x;\beta,\theta\, dx$$

$$m_r(t) = \frac{\theta}{\beta^{k+2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \int_0^t x^{k+r+1} e^{-x\left(\frac{\theta i+\theta+j}{\beta}\right)} dx$$

In simpler terms

$$m_r(t) = \frac{\theta\beta^r}{(\theta(i+1)+j)^{r+k+2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \int_0^{\frac{t}{\beta}\theta(i+1)+j} y^{(r+k+2)-1}\ e^{-y} dy$$

Using the incomplete gamma function

$$m_r(t) = \frac{\theta\beta^r}{(\theta(i+1)+j)^{r+k+2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \ .\gamma\left(\mathrm{r}+\mathrm{k}+2, \frac{t}{\beta}\theta(i+1)+j\right).$$

where, $\gamma\left(\mathrm{r}+\mathrm{k}+2, \frac{t}{\beta}\theta(i+1)+j\right)$ is the lower incomplete Gamma function.

The first ICM is obtained by replacing r with 1 in the r-th incomplete moment, and the analytical expression for ICM is given as

$$m_1(t) = \frac{\theta\beta}{(\theta(i+1)+j)^{k+3}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \ .\gamma\left(\mathrm{k}+3, \frac{t}{\beta}\theta(i+1)+j\right).$$

3.1.2 Bonferroni and Lorenz Curve

The Bonferroni and Lorenz curves are significant tools that find a lot of applications in economics, demography, insurance, income distribution, and poverty analysis to examine property of inequality and concentration of a distribution. These curves are the cumulative proportions of a variable of the cumulative proportions of the population.

For a random variable X with mean μ, the Bonferroni curve $B_f(q)$and Lorenz Curve $L_r(q)$ curves are respectively defined as

$$B_f(q) = \frac{1}{q\mu}\int_0^p xf(x) \qquad ; \qquad L_r(q) = \frac{1}{\mu}\int_0^p xf(x)$$

Using the first ICM, the Bonferroni and Lorenz Curves are provided as follows, respectively.

$$B_f(q) = \frac{1}{q\mu}\left[\frac{\theta\beta}{(\theta(i+1)+j)^{k+3}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk}\, \gamma\left(k+3, \frac{p}{\beta}\theta(i+1)+j\right)\right]$$

$$B_f(q) = \frac{1}{\mu}\left[\frac{\theta\beta}{(\theta(i+1)+j)^{k+3}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk}\, \gamma\left(k+3, \frac{p}{\beta}\theta(i+1)+j\right)\right]$$

These expressions provide analytical forms for the Bonferroni and Lorenz curves of the GFME distribution, which can be used to study inequality and concentration characteristics associated with the distribution.

3.2 Generating functions

Theorem 3: *If X ~ GFME* $(x;\theta,\beta)$ *with scale* $(\beta > 0)$, *shape* $(\theta > 0)$ *parameters, then the moment generating function* $(m_x(t))$ *of X is given by*

$$m_x(t) = \left[\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\varphi_{ijk}\Gamma(k+2)}{(\theta(i+1)+j-\beta t)^{k+2}}\right].$$

Proof: Mathematically, the moment generating function is given as

$$m_x(t) = \int_0^{\infty} e^{tx} f|x;\beta,\theta\, dx$$

Using equation (7)

$$m_x(t) = \int_0^{\infty} e^{tx} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \ \frac{\theta}{\beta^{k+2}} x^{k+1} e^{-x\left(\frac{\theta i+\theta+j}{\beta}\right)} dx$$

Then the analytical expression of the MGF of X is written as follows:

$$m_x(t) = \frac{\theta}{\beta^{2+k}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \int_0^{\infty} x^{k+1}\ e^{-\frac{x}{\beta}(\theta(i+1)+j-\beta t)} dx$$

After simplification,

$$m_x(t) = \frac{\theta}{(\theta(i+1)+j-\beta t\,)^{k+2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varphi_{ijk} \int_0^{\infty} y^{(k+2)-1}\ e^{-y} dy$$

Using the Gamma function,

$$m_x(t) = \left[\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varphi_{ijk} \Gamma(k+2)}{(\theta(i+1) + j - \beta t)^{k+2}} \right].$$

Similarly, the characteristic function and cumulant generating function of the proposed distribution can be derived by means of the moment generating function. The characteristic function of a random variable X is obtained by following the same procedure used for the moment generating function and replacing t with it. The characteristic function of $X \sim GFME(\theta, \beta)$ is obtained as

$$\varphi_x(t) = \left[\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varphi_{ijk} \Gamma(k+2)}{(\theta(i+1) + j - \beta it)^{k+2}} \right].$$

Further, the cumulant generating function (CGF) is defined as the natural logarithm of the moment generating function. Thus, using the expression of the moment generating function derived in Theorem 3, the cumulant generating function of the GFME distribution is given by

$$K_x(t) = log \left[\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varphi_{ijk} \Gamma(k+2)}{(\theta(i+1) + j - \beta it)^{k+2}} \right].$$

These functions are useful in obtaining higher-order moments and cumulants of the GFME distribution.

3.3 Quantile function and related measures

The quantile function of the GFME distribution is obtained by inverting its cumulative distribution function. Let $u = F(x; \beta, \theta)$, where $0 < u < 1$. The CDF of the GFME distribution is given by

$$F(x; \beta, \theta) = \exp\left[-\left(\frac{\left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}}{1 - \left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}} \right)^{\theta} \right]$$

Setting $u = F(x; \beta, \theta$ *yeilds*

$$\exp\left[-\left(\frac{\left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}}{1 - \left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}} \right)^{\theta} \right] = u$$

Taking the natural logarithm on both sides and simplifying for x gives

$$\left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}} \left(1 + (-ln(u))^{\frac{1}{\theta}}\right) = (-ln(u))^{\frac{1}{\theta}}$$

To isolate x, the above equation is expressed in a form suitable for the Lambert W function. Multiplying and rearranging terms leads to

$$-\left(1 + \frac{x}{\beta}\right) e^{-\left(\frac{x}{\beta} + 1\right)} = -\frac{(-ln(u))^{\frac{1}{\theta}}}{e\left(1 + (-ln(u))^{\frac{1}{\theta}}\right)}$$

Using the identity of the Lambert W function, $-\mathrm{W(z)e^{-W(z)}} = -\mathrm{z}$, we obtain

$$-\left(1 + \frac{x}{\beta}\right) = W\left(-\frac{(-ln(u))^{\frac{1}{\theta}}}{e\left(1 + (-ln(u))^{\frac{1}{\theta}}\right)} \right)$$

Hence,

$$Q(u) = \beta \left[-W\left(-\frac{\left(-ln(u)\right)^{\frac{1}{\theta}}}{e\left(1 + \left(-ln(u)\right)^{\frac{1}{\theta}}\right)} \right) - 1 \right]$$

Here, $W(\cdot)$ denotes the Lambert W function. If u follows a uniform distribution on (0,1), the above expression can be used to generate random observations from the GFME distribution using the inverse transformation method.

Based on the quantile function, several quantile-based descriptive measures can also be defined. In particular, the Bowley coefficient of skewness ($\Lambda_{\mathrm{B-sk}}$) and the Moors coefficient of kurtosis (Λ_{M-kur}) provide robust measures of asymmetry and tail behavior of the distribution. These measures are defined respectively as

$$\mathit{\Lambda}_B - sk = \frac{qf|0.75 + qf|0.25 - 2qf|0.50}{qf|0.75 - qf|0.25}$$

And

$$\Lambda_M - kur = \frac{O|0.375 - O|0.125 - O|0.625 + O|0.875}{q|0.75 - q|0.25}$$

These quantile-based measures are particularly useful because they are less sensitive to extreme observations and provide additional insight into the shape characteristics of the GFME distribution**.**

3.4 Entropies

In the following section, a few entropy values of the GFME distribution are addressed. Entropy measures are relevant in the field of information theory and statistics because these measures are used to measure the uncertainty or randomness of a probability distribution. In the case of the suggested GFME distribution, various entropy values like the Rényi entropy, generalized entropy, q- entropy, Tsallis entropy, Havrda-Charvat entropy, and generalized Havrda-Charvat entropy are obtained.

Let X be a random variable following the GFME distribution with PDF $f(x)$. The Rényi entropy of order $\delta(\delta > 0,\ \delta \neq 1)$ is defined as

$$H_\delta(X) = \frac{1}{1-\delta}\log\left[\int_0^\infty f(x;\beta,\theta)^\delta dx\right]$$

For the Rényi entropy (RE), we recall the probability density function given in Equation (7). Thus,

$$f(x;\beta,\theta)^\delta = \int_0^\infty \left[\frac{\theta^\delta \frac{x^\delta}{\beta^{2\delta}} e^{-\frac{\delta x}{\beta}}\left(\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\delta(\theta-1)}}{(1+x/\beta)e^{-\frac{x}{\beta}})^{\delta(\theta+1)}}\right]\exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\delta\theta}\right]dx,$$

Using the exponential and binomial series expansions, the above expression can be written as

$$f(x;\beta,\theta)^\delta = \frac{\theta^\delta}{\beta^{2\delta+k}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\int_0^\infty x^{\delta+k}\ (e^{-\frac{x}{\beta}})^{(\delta\theta+\delta\theta i)+j}\ dx,$$

Evaluating the integral and simplifying the resulting expression, we obtain

$$f(x;\beta,\theta)^\delta = \left[\frac{\theta^\delta}{\beta^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)\right].$$

Substituting this result into the definition of Rényi entropy yields

$$H_\delta(X) = \frac{1}{1-\delta}\log\left[\frac{\theta^\delta}{\beta^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)\right].$$

Similarly, the generalized Rényi entropy of order δ is given by

$$GH_\delta(X) = \frac{1}{1-\delta}\log\left[\frac{\theta^\delta}{(\alpha\beta)^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)\right].$$

The Tsallis (1988) Entropy is given as

$$T_R(\delta) = \frac{1}{\delta-1}\left[1-\frac{\theta^\delta}{\beta^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)\right]$$

The q-Entropy (H_q) is given as

$$H_q = \frac{1}{q-1}\log\left[1-(1-q)\left(\frac{\theta^\delta}{\beta^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)\right)\right]$$

The Havrda and Charvat’s entropy is given as

$$HC(x) = \frac{1}{2^{\delta-1}-1}\left[\frac{\theta^\delta}{\beta^{\delta-1}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)-1\right]$$

Finally, the generalized Havrda and Charvat’s entropy can be written as

$$GHC(x) = \frac{1}{2^{1-\delta}-1}\left[\frac{\theta^\delta}{(\alpha\beta)^{\delta-1}}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{ijk}\frac{1}{(\delta\theta+\delta\theta i+j)^{k+\delta+1}}\Gamma(\delta+k+1)\right)-1\right]$$

3.5 Order Statistics

Order statistics (OS) is a key topic in the argument of inferential and non-parametric statistics. It has shown its value in both theoretical and practical disciplines. Let's suppose that the $X|_1,\dots,X|_n$ is a size *n* random sample drawn from the order statistics (OS) $X|_{(1:n)} < \dots < X|_{(n:n)}$. The OS PDF is written as

$$f_{X_{(k)}} = \frac{1}{\beta(k,n-k+1)}\,p(x)[\,F(x)]^{k-1}[1-F(x)]^{n-k} \qquad k = 1,2,3,\dots,n$$

The OS PDF for the GFME distribution is obtained by placing (5) and (6) in the above equation. The kth OS is given as follows.

$$f_{X_{(k)}}=\frac{1}{\beta(k,n-k+1)}\left[\begin{array}{c}\left(\frac{\theta\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left(\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}}\exp\left\{-\left[\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right]^{\theta}\right\}\right)\\ \left(Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right)^{k-1}\left(1-Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right)^{n-k}\end{array}\right]$$

The first minimum OS (min-OS) of the sample is defined as $f_{x|min\text{-}OS}=min\{X|_{(1:n)}<\ \ldots<X|_{(n:n)}\}$. For convenience, it can be presented as follows:

$$f_{X_{(1)}}=np(x)[1-F(x)]^{n-1}$$

$$f_{X_{(1)}}=n\left[\begin{array}{c}\left(\frac{\theta\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left(\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}}\exp\left\{-\left[\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right]^{\theta}\right\}\right)\\ \left(1-Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right)^{n-1}\end{array}\right]$$

The maximum OS (max-OS) of the sample is defined as $f_{x|max\text{-}OS}=min\{X|_{(1:n)}<\ \ldots<X|_{(n:n)}\}$. For convenience, it can be presented as follows:

$$f_{X_{(1)}}=np(x)[F(x)]^{n-1}$$

For the GFME distribution, it is written as

$$f_{X_{(n)}}=n\left[\begin{array}{c}\left(\frac{\theta\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\left(\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta-1}}{\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{\theta+1}}\exp\left\{-\left[\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right]^{\theta}\right\}\right)\\ \left(Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right)^{n-1}\end{array}\right]$$

The cumulative distribution function of the jth order statistic can be obtained by applying the following formula,

$$F_{X_{(j)}}=\sum_{k=j}^{n}\binom{n}{k}[F(x)]^{k}[1-F(x)]^{n-k}$$

The j-th OS cdf for the GFME distribution is written as

$$F_{X_{(j)}}=\sum_{k=j}^{n}\binom{n}{k}\left[Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right]^{k}\left[1-Exp\left[-\left(\frac{\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}{1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}}\right)^{\theta}\right]\right]^{n-k}.$$

4. STATISTICAL INFERENCE AND SIMULATION

In this section, some of the classical methods of estimation are used to get a parameter estimate. Such techniques are maximum likelihood estimation, Anderson-Darling estimation, Cramer-von Mises estimation, maximum product of spacings estimation, least squares estimation, and weighted least squares estimation. Each approach estimates the parameters by optimizing objective functions based on the probability density function and cumulative distribution function of the intended distribution. The methods of the estimation of the parameters of the GFME distribution are provided in the following subsections.

4.1. Maximum likelihood

Let x_1, x_2,..., x_n denote a random sample from the GFME distribution with parameters θ and β. The log-likelihood function for the sample can be written as:

$$ln(\Delta) = nln\theta + \sum_{i=1}^{n} lnx_i - 2nln\beta - \sum_{i=1}^{n} \frac{x_i}{\beta} + (\theta - 1)\sum_{i=1}^{n} ln\left(\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right)$$
$$- (\theta + 1)\sum_{i=1}^{n} ln\left(1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right) - \sum_{i=1}^{n}\left(\frac{\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right)^{\theta}$$

To obtain the maximum likelihood estimators (MLEs), we differentiate the log-likelihood with respect to the parameters and set it to zero. The first derivative with respect to θ is:

$$\frac{\partial ln(\Delta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln\left(\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right) - \sum_{i=1}^{n} ln\left(1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right)$$
$$- \sum_{i=1}^{n}\left(\frac{\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right)^{\theta} ln\left(\frac{\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right)$$

The derivative with respect to β is:

$$\frac{\partial ln(\Delta)}{\partial\theta} = -\frac{2n}{\beta} + \sum_{i=1}^{n}\frac{x_i}{\beta^2} + (\theta - 1)\sum_{i=1}^{n}\frac{x^2 e^{-\frac{x_i}{\beta}}}{\beta^3\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}} + (\theta + 1)\sum_{i=1}^{n}\frac{x^2 e^{-\frac{x_i}{\beta}}}{\beta^3\left(1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right)}$$
$$- \theta\sum_{i=1}^{n}\left[\frac{\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right]^{\theta - 1}\left[\frac{x^2 e^{-\frac{x_i}{\beta}}}{\beta^3\left(1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}\right)^2}\right]$$

The MLEs of θ and β are obtained by solving $\frac{\partial L}{\partial\theta} = 0$and $\frac{\partial L}{\partial\beta} = 0$ simultaneously. Due to the nonlinear and complex nature of these equations, closed-form solutions are not available. Numerical optimization methods, such as the Newton–Raphson algorithm, are typically used to obtain parameter estimates. These MLEs are asymptotically efficient and unbiased under standard regularity conditions.

4.2. Least Squares (LS) and Weighted Least Squares (WLS)

Following Swain et al. (1988), the least squares (LS) method estimates the parameters by minimizing the sum of squared differences between the theoretical cumulative distribution function (CDF) $F(x)$ and the empirical probabilities**:**

$$LS(\Delta) = \sum_{i=1}^{n}\left(F(x_i) - \frac{i}{n + 1}\right)^2 = \sum_{i=1}^{n}\left(Exp\left[-\left(\frac{\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right)^{\theta}\right] - \frac{i}{n + 1}\right)^2.$$

The weighted least squares (WLS) method assigns weights to each term to account for variability across the order statistics:

$$WLS\ (\Delta)\ =\ \sum_{i=1}^{n}\frac{((n + 1)^2)(n + 2)}{i(n - i + 1)}\left(F(x_i) - \frac{i}{n + 1}\right)^2$$

Parameter estimates are obtained by solving the nonlinear equations $\left(\frac{\partial \mathrm{WLS}}{\partial\theta} and\ \frac{\partial \mathrm{WLS}}{\partial\beta}\right)^T = 0$, numerically.

4.3. Cramer–von Mises (CVM) Method

The Cramer–von Mises approach minimizes the discrepancy between the empirical distribution function and the theoretical CDF.

$$CVM\ (\Delta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_i) - \frac{2i - 1}{2n}\right)^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left(Exp\left[-\left(\frac{\left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1 - \left(1 + \frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right)^{\theta}\right] - \frac{2i - 1}{2n}\right)^2$$

4.4. Anderson–Darling (AD) Method

The Anderson–Darling method emphasizes the tails of the distribution by minimizing a weighted squared difference between the empirical and theoretical CDFs:

$$AD\ (\Delta)\ =\ -n - \frac{1}{n}\sum_{i=1}^{n}(2i - 1)\left(logF(x_i) - logS(x_{n+1-i})\right)$$

Where $S(x) = 1 - F(x)$ is survival function. For the FGME distribution, this becomes:

$$AD(\Delta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left(Exp\left[-\left(\frac{\left(1+\frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}{1-\left(1+\frac{x_i}{\beta}\right)e^{-\frac{x_i}{\beta}}}\right)^{\theta}\right]\right)\right.$$
$$\left. - \log\left(1-Exp\left[-\left(\frac{\left(1+\frac{x_{n+1-i}}{\beta}\right)e^{-\frac{x_{n+1-i}}{\beta}}}{1-\left(1+\frac{x_{n+1-i}}{\beta}\right)e^{-\frac{x_{n+1-i}}{\beta}}}\right)^{\theta}\right]\right)\right)$$

All these methods require solving nonlinear equations numerically to obtain parameter estimates. They provide alternatives to maximum likelihood estimation and may be preferable in cases where MLE is difficult to compute or when emphasizing different parts of the distribution, such as the tails.

4.5. Simulation Study

A Monte Carlo simulation experiment was performed to assess the performance of MLE, ADE, CVME, OLSE, and WLSE. A few combinations of the parameters (θ, β) were taken into account. The sample sizes were $n = 25, 50, 100, 150, 200$ and 300. To each pair of (θ, β) and n observations were produced based on the GFME distribution. MLE, ADE, CVME, OLSE, and WLSE were then used to obtain the parameters. The repetition of this process was 10000 times. The performance of the estimators was measured in terms of Average Estimate (AE), Absolute Bias (AB), Mean Relative error (MRE) and Mean Squared error (MSE). The formulas of these measures are presented below.

$$AB(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}\left|\hat{\beta}-\beta\right| \quad \& \quad AB(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}\left|\hat{\theta}-\theta\right|,$$

$$MRE(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}\frac{\left|\hat{\beta}-\beta\right|}{\beta} \quad \& \quad MRE(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}\frac{\left|\hat{\theta}-\theta\right|}{\theta},$$

and

$$MSE(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{\beta}-\beta\right)^2 \quad \& \quad MSE(\hat{\theta}) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{\theta}-\theta\right)^2.$$

All the simulation findings are presented in Tables 2-10.

Table 2. Estimates of the parameters for the GFME model with $\theta = 0.5, \theta = 0.5$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	0.53356	0.51133	0.5398	0.50286	0.50902
		AB	0.03356	0.01133	0.0398	0.00286	0.00902
		MRE	0.06713	0.02266	0.07959	0.00571	0.01803
		MSE	0.00911	0.00845	0.01682	0.01267	0.01102
	β	AE	0.51147	0.50644	0.51347	0.50514	0.50458
		AB	0.01147	0.00644	0.01347	0.00514	0.00458
		MRE	0.02293	0.01288	0.02695	0.01028	0.00915
		MSE	0.00683	0.00797	0.01014	0.00952	0.00915
50	θ	AE	0.51729	0.50234	0.51666	0.49923	0.5067
		AB	0.01729	0.00234	0.01666	0.00077	0.0067
		MRE	0.03459	0.00467	0.03332	0.00155	0.0134
		MSE	0.00361	0.00358	0.00621	0.00542	0.00443
	β	AE	0.50536	0.50466	0.50719	0.50407	0.50405
		AB	0.00536	0.00466	0.00719	0.00407	0.00405
		MRE	0.01072	0.00932	0.01439	0.00815	0.0081
		MSE	0.00329	0.00409	0.00509	0.00499	0.00407
100	θ	AE	0.50727	0.50411	0.51074	0.50203	0.50286
		AB	0.00727	0.00411	0.01074	0.00203	0.00286
		MRE	0.01454	0.00823	0.02148	0.00407	0.00572
		MSE	0.00156	0.00196	0.00289	0.0027	0.00205
	β	AE	0.5027	0.50261	0.50446	0.50212	0.50173
		AB	0.0027	0.00261	0.00446	0.00212	0.00173
		MRE	0.00539	0.00522	0.00893	0.00423	0.00346
		MSE	0.00157	0.00202	0.00258	0.00246	0.0019
150	θ	AE	0.50493	0.50215	0.50553	0.50007	0.50258
		AB	0.00493	0.00215	0.00553	0.00007	0.00258
		MRE	0.00985	0.0043	0.01107	0.00015	0.00516
		MSE	0.00101	0.00117	0.00168	0.00156	0.00129

	β	AE	0.50168	0.50168	0.50249	0.50118	0.50174
		AB	0.00168	0.00168	0.00249	0.00118	0.00174
		MRE	0.00336	0.00336	0.00497	0.00236	0.00348
		MSE	0.00102	0.00132	0.00158	0.00158	0.00139
200	θ	AE	0.50331	0.50116	0.50462	0.49991	0.50007
		AB	0.00331	0.00116	0.00462	0.00009	0.00007
		MRE	0.00661	0.00233	0.00923	0.00017	0.00014
		MSE	0.00075	0.0009	0.0013	0.00124	0.00091
	β	AE	0.50125	0.49937	0.50015	0.49905	0.49926
		AB	0.00125	0.00063	0.00015	0.00095	0.00074
		MRE	0.00249	0.00126	0.0003	0.00189	0.00149
		MSE	0.00077	0.00095	0.00116	0.00115	0.00098
300	θ	AE	0.50288	0.50034	0.50222	0.49936	0.50108
		AB	0.00288	0.00034	0.00222	0.00064	0.00108
		MRE	0.00577	0.00069	0.00445	0.00128	0.00216
		MSE	0.0005	0.00062	0.00088	0.00084	0.00065
	β	AE	0.5013	0.4999	0.50003	0.4995	0.49983
		AB	0.0013	0.0001	0.00003	0.0005	0.00017
		MRE	0.00261	0.0002	0.00006	0.00099	0.00035
		MSE	0.0005	0.00067	0.00085	0.00081	0.00065

Table 3. Estimates of the parameters for the GFME model with $\theta = 0.5, \theta = 1.0$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	0.53222	0.51049	0.5398	0.50328	0.50902
		AB	0.03222	0.01049	0.0398	0.00328	0.00902
		MRE	0.06443	0.02098	0.07959	0.00656	0.01803
		MSE	0.01641	0.00817	0.01682	0.01348	0.01102
	β	AE	0.97464	1.01641	1.02695	1.01169	1.00915
		AB	0.02536	0.01641	0.02695	0.01169	0.00915
		MRE	0.02536	0.01641	0.02695	0.01169	0.00915
		MSE	0.07612	0.03089	0.04055	0.03821	0.03662
50	θ	AE	0.51658	0.50727	0.51763	0.50373	0.5067
		AB	0.01658	0.00727	0.01763	0.00373	0.0067
		MRE	0.03316	0.01454	0.03526	0.00745	0.0134
		MSE	0.01127	0.00382	0.00623	0.00564	0.00443
	β	AE	0.95997	1.0059	1.00963	1.00378	1.0081
		AB	0.04003	0.0059	0.00963	0.00378	0.0081
		MRE	0.04003	0.0059	0.00963	0.00378	0.0081
		MSE	0.06617	0.01595	0.0205	0.01971	0.01628
100	θ	AE	0.50809	0.50293	0.51076	0.50103	0.50286
		AB	0.00809	0.00293	0.01076	0.00103	0.00286
		MRE	0.01617	0.00586	0.02152	0.00206	0.00572
		MSE	0.00849	0.00173	0.00283	0.00246	0.00205
	β	AE	0.95872	1.00519	1.0112	1.0048	1.00346
		AB	0.04128	0.00519	0.0112	0.0048	0.00346
		MRE	0.04128	0.00519	0.0112	0.0048	0.00346
		MSE	0.05356	0.00811	0.01003	0.00989	0.0076
150	θ	AE	0.50529	0.50102	0.50488	0.49952	0.50258
		AB	0.00529	0.00102	0.00488	0.00048	0.00258
		MRE	0.01057	0.00205	0.00977	0.00096	0.00516
		MSE	0.00859	0.00126	0.0017	0.00167	0.00129
	β	AE	0.95351	0.99988	1.00047	0.99911	1.00348
		AB	0.04649	0.00012	0.00047	0.00089	0.00348
		MRE	0.04649	0.00012	0.00047	0.00089	0.00348
		MSE	0.05348	0.00546	0.00595	0.00653	0.00557
200	θ	AE	0.5045	0.50075	0.50363	0.49913	0.50007
		AB	0.0045	0.00075	0.00363	0.00087	0.00007
		MRE	0.00901	0.0015	0.00727	0.00174	0.00014
		MSE	0.00831	0.00085	0.00119	0.00117	0.00091
	β	AE	0.95252	1.00487	1.00673	1.00473	0.99851
		AB	0.04748	0.00487	0.00673	0.00473	0.00149
		MRE	0.04748	0.00487	0.00673	0.00473	0.00149
		MSE	0.05109	0.00347	0.00413	0.00413	0.00392

300	θ	AE	0.50374	0.50036	0.50244	0.49937	0.50108
		AB	0.00374	0.00036	0.00244	0.00063	0.00108
		MRE	0.00748	0.00072	0.00487	0.00127	0.00216
		MSE	0.00739	0.00062	0.00086	0.00084	0.00065
	β	AE	0.95608	1.00177	1.00216	1.00109	0.99965
		AB	0.04392	0.00177	0.00216	0.00109	0.00035
		MRE	0.04392	0.00177	0.00216	0.00109	0.00035
		MSE	0.0474	0.00255	0.00325	0.00306	0.00261

Table 4. Estimates of the parameters for the GFME model with $\theta = 0.5, \theta = 1.5$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	0.53322	0.51133	0.5398	0.50286	0.50902
		AB	0.03322	0.01133	0.0398	0.00286	0.00902
		MRE	0.06644	0.02266	0.07959	0.00571	0.01803
		MSE	0.00877	0.00845	0.01682	0.01267	0.01102
	β	AE	1.53128	1.51932	1.54042	1.51543	1.51373
		AB	0.03128	0.01932	0.04042	0.01543	0.01373
		MRE	0.02085	0.01288	0.02695	0.01028	0.00915
		MSE	0.06071	0.07175	0.09125	0.08571	0.08239
50	θ	AE	0.51595	0.50234	0.51666	0.49923	0.5067
		AB	0.01595	0.00234	0.01666	0.00077	0.0067
		MRE	0.0319	0.00467	0.03332	0.00155	0.0134
		MSE	0.00361	0.00358	0.00621	0.00542	0.00443
	β	AE	1.51249	1.51398	1.52158	1.51222	1.51214
		AB	0.01249	0.01398	0.02158	0.01222	0.01214
		MRE	0.00832	0.00932	0.01439	0.00815	0.0081
		MSE	0.03014	0.0368	0.04582	0.04493	0.03663
100	θ	AE	0.50816	0.50411	0.51074	0.50203	0.50286
		AB	0.00816	0.00411	0.01074	0.00203	0.00286
		MRE	0.01632	0.00823	0.02148	0.00407	0.00572
		MSE	0.00189	0.00196	0.00289	0.0027	0.00205
	β	AE	1.5044	1.50783	1.51339	1.50635	1.50519
		AB	0.0044	0.00783	0.01339	0.00635	0.00519
		MRE	0.00293	0.00522	0.00893	0.00423	0.00346
		MSE	0.01968	0.01818	0.0232	0.02211	0.01711
150	θ	AE	0.50586	0.50215	0.50553	0.50007	0.50258
		AB	0.00586	0.00215	0.00553	0.00007	0.00258
		MRE	0.01172	0.0043	0.01107	0.00015	0.00516
		MSE	0.00169	0.00117	0.00168	0.00156	0.00129
	β	AE	1.49485	1.50504	1.50746	1.50353	1.50522
		AB	0.00515	0.00504	0.00746	0.00353	0.00522
		MRE	0.00343	0.00336	0.00497	0.00236	0.00348
		MSE	0.01971	0.01188	0.01424	0.01419	0.01253
200	θ	AE	0.50316	0.50116	0.50461	0.49991	0.50007
		AB	0.00316	0.00116	0.00461	0.00009	0.00007
		MRE	0.00632	0.00233	0.00923	0.00017	0.00014
		MSE	0.00154	0.0009	0.0013	0.00124	0.00091
	β	AE	1.48996	1.49811	1.50045	1.49716	1.49777
		AB	0.01004	0.00189	0.00045	0.00284	0.00223
		MRE	0.00669	0.00126	0.0003	0.00189	0.00149
		MSE	0.02023	0.00857	0.01041	0.01032	0.00882
300	θ	AE	0.50249	0.50034	0.50222	0.49936	0.50108
		AB	0.00249	0.00034	0.00222	0.00064	0.00108
		MRE	0.00497	0.00069	0.00445	0.00128	0.00216
		MSE	0.00172	0.00062	0.00088	0.00084	0.00065
	β	AE	1.48385	1.4997	1.5001	1.49851	1.49948
		AB	0.01615	0.0003	0.0001	0.00149	0.00052
		MRE	0.01077	0.0002	0.00006	0.00099	0.00035
		MSE	0.02363	0.00601	0.00763	0.00726	0.00588
		MRE	0.01349	0.00096	0.00018	0.00147	0.00029
		MSE	0.02701	0.00457	0.00588	0.00549	0.00477

Table 5. Estimates of the parameters for the GFME model with $\theta = 0.5, \theta = 2.0$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	0.53348	0.51049	0.5398	0.50328	0.50902
		AB	0.03348	0.01049	0.0398	0.00328	0.00902
		MRE	0.06695	0.02098	0.0796	0.00656	0.01804
		MSE	0.009	0.00817	0.01682	0.01348	0.01102
	β	AE	2.04761	2.03282	2.0539	2.02338	2.0183
		AB	0.04761	0.03282	0.0539	0.02338	0.0183
		MRE	0.02381	0.01641	0.02695	0.01169	0.00915
		MSE	0.10799	0.12356	0.16222	0.15285	0.14646
50	θ	AE	0.51585	0.50727	0.51763	0.50373	0.5067
		AB	0.01585	0.00727	0.01763	0.00373	0.0067
		MRE	0.0317	0.01454	0.03526	0.00745	0.0134
		MSE	0.00355	0.00382	0.00623	0.00564	0.00443
	β	AE	2.02023	2.01181	2.01925	2.00757	2.01619
		AB	0.02023	0.01181	0.01925	0.00757	0.01619
		MRE	0.01011	0.0059	0.00963	0.00378	0.0081
		MSE	0.05135	0.0638	0.08201	0.07882	0.06511
100	θ	AE	0.50748	0.50293	0.51076	0.50103	0.50286
		AB	0.00748	0.00293	0.01076	0.00103	0.00286
		MRE	0.01496	0.00586	0.02152	0.00206	0.00572
		MSE	0.00162	0.00173	0.00283	0.00246	0.00205
	β	AE	2.01208	2.01039	2.02241	2.00959	2.00691
		AB	0.01208	0.01039	0.02241	0.00959	0.00691
		MRE	0.00604	0.00519	0.01121	0.0048	0.00346
		MSE	0.02576	0.03243	0.04012	0.03955	0.03041
150	θ	AE	0.50564	0.50102	0.50488	0.49952	0.50258
		AB	0.00564	0.00102	0.00488	0.00048	0.00258
		MRE	0.01129	0.00205	0.00977	0.00096	0.00516
		MSE	0.00105	0.00126	0.0017	0.00167	0.00129
	β	AE	2.00497	1.99976	2.00094	1.99821	2.00695
		AB	0.00497	0.00024	0.00094	0.00179	0.00695
		MRE	0.00248	0.00012	0.00047	0.00089	0.00348
		MSE	0.01686	0.02184	0.02382	0.02613	0.02227
200	θ	AE	0.50416	0.50075	0.50363	0.49913	0.50007
		AB	0.00416	0.00075	0.00363	0.00087	0.00007
		MRE	0.00833	0.0015	0.00727	0.00174	0.00014
		MSE	0.00076	0.00085	0.00119	0.00117	0.00091
	β	AE	2.00442	2.00974	2.01346	2.00947	1.99703
		AB	0.00442	0.00974	0.01346	0.00947	0.00297
		MRE	0.00221	0.00487	0.00673	0.00473	0.00149
		MSE	0.01282	0.0139	0.01652	0.0165	0.01569
300	θ	AE	0.50204	0.50036	0.50244	0.49937	0.50108
		AB	0.00204	0.00036	0.00244	0.00063	0.00108
		MRE	0.00408	0.00072	0.00487	0.00127	0.00216
		MSE	0.00051	0.00062	0.00086	0.00084	0.00065
	β	AE	2.00201	2.00355	2.00431	2.00218	1.9993
		AB	0.00201	0.00355	0.00431	0.00218	0.0007
		MRE	0.001	0.00177	0.00216	0.00109	0.00035
		MSE	0.00865	0.01021	0.01302	0.01224	0.01045

Table 6. Estimates of the parameters for the GFME model with $\theta = 1.0$, $\theta = 0.5$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	1.06133	1.01995	1.06562	0.99799	1.00705
		AB	0.06133	0.01995	0.06562	0.00201	0.00705
		MRE	0.06133	0.01995	0.06562	0.00201	0.00705
		MSE	0.03736	0.02972	0.062	0.04016	0.03687
	β	AE	0.50281	0.50503	0.50742	0.50286	0.49977
		AB	0.00281	0.00503	0.00742	0.00286	0.00023
		MRE	0.00563	0.01006	0.01483	0.00572	0.00046
		MSE	0.00324	0.00255	0.00305	0.00272	0.00225
		AE	1.02636	1.00647	1.02878	0.99696	1.00931
		AB	0.02636	0.00647	0.02878	0.00304	0.00931

50	θ	MRE	0.02636	0.00647	0.02878	0.00304	0.00931
		MSE	0.01719	0.01439	0.02249	0.01977	0.01666
	β	AE	0.50021	0.50218	0.50289	0.50144	0.50327
		AB	0.00021	0.00218	0.00289	0.00144	0.00327
		MRE	0.00042	0.00436	0.00577	0.00288	0.00654
		MSE	0.00215	0.00122	0.0014	0.00135	0.0013
100	θ	AE	1.01155	1.00447	1.01423	0.99917	1.00384
		AB	0.01155	0.00447	0.01423	0.00083	0.00384
		MRE	0.01155	0.00447	0.01423	0.00083	0.00384
		MSE	0.00986	0.00687	0.00959	0.00895	0.0067
	β	AE	0.49923	0.50121	0.5024	0.50064	0.5002
		AB	0.00077	0.00121	0.0024	0.00064	0.0002
		MRE	0.00153	0.00241	0.0048	0.00128	0.00041
		MSE	0.00148	0.00054	0.00062	0.0006	0.00058
150	θ	AE	1.00571	1.00228	1.01123	0.99849	1.0018
		AB	0.00571	0.00228	0.01123	0.00151	0.0018
		MRE	0.00571	0.00228	0.01123	0.00151	0.0018
		MSE	0.0089	0.00446	0.00652	0.0059	0.00437
	β	AE	0.4979	0.50075	0.5011	0.50045	0.50069
		AB	0.0021	0.00075	0.0011	0.00045	0.00069
		MRE	0.00419	0.0015	0.00219	0.00091	0.00138
		MSE	0.00162	0.00039	0.00044	0.00042	0.00038
200	θ	AE	1.00197	1.00421	1.0115	1.0023	0.99927
		AB	0.00197	0.00421	0.0115	0.0023	0.00073
		MRE	0.00197	0.00421	0.0115	0.0023	0.00073
		MSE	0.00804	0.00357	0.00506	0.00476	0.00333
	β	AE	0.49776	0.50032	0.50109	0.50025	0.50122
		AB	0.00224	0.00032	0.00109	0.00025	0.00122
		MRE	0.00448	0.00064	0.00219	0.00049	0.00244
		MSE	0.00155	0.00032	0.00036	0.00036	0.00029
300	θ	AE	0.99945	1.00323	1.00662	1.00105	1.00224
		AB	0.00055	0.00323	0.00662	0.00105	0.00224
		MRE	0.00055	0.00323	0.00662	0.00105	0.00224
		MSE	0.00723	0.00221	0.00294	0.0029	0.00233
	β	AE	0.4976	0.49995	0.50014	0.49968	0.50018
		AB	0.0024	0.00005	0.00014	0.00032	0.00018
		MRE	0.0048	0.00011	0.00028	0.00064	0.00036
		MSE	0.00148	0.00019	0.00021	0.0002	0.00019

Table 7. Estimates of the parameters for the GFME model with $\theta = 1.5$, $\theta = 0.5$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	1.59029	1.52779	1.59757	1.5021	1.51869
		AB	0.09029	0.02779	0.09757	0.0021	0.01869
		MRE	0.0602	0.01853	0.06505	0.0014	0.01246
		MSE	0.0767	0.07338	0.1259	0.09912	0.08261
	β	AE	0.50289	0.50251	0.50309	0.50095	0.50039
		AB	0.00289	0.00251	0.00309	0.00095	0.00039
		MRE	0.00577	0.00502	0.00617	0.0019	0.00079
		MSE	0.00141	0.00118	0.00126	0.00125	0.00122
50	θ	AE	1.54166	1.51847	1.54803	1.50206	1.51521
		AB	0.04166	0.01847	0.04803	0.00206	0.01521
		MRE	0.02777	0.01231	0.03202	0.00138	0.01014
		MSE	0.03204	0.03437	0.05189	0.0462	0.03576
	β	AE	0.50077	0.5009	0.50187	0.50005	0.5009
		AB	0.00077	0.0009	0.00187	0.00005	0.0009
		MRE	0.00155	0.00179	0.00373	0.00011	0.0018
		MSE	0.00089	0.00055	0.0006	0.00059	0.00057
100	θ	AE	1.52198	1.51134	1.52695	1.50409	1.50715
		AB	0.02198	0.01134	0.02695	0.00409	0.00715
		MRE	0.01466	0.00756	0.01796	0.00273	0.00477
		MSE	0.01489	0.01629	0.02315	0.02159	0.01791
	β	AE	0.50002	0.49915	0.49951	0.49879	0.50038
		AB	0.00002	0.00085	0.00049	0.00121	0.00038

		MRE	0.00004	0.0017	0.00098	0.00243	0.00076
		MSE	0.00055	0.00025	0.00028	0.00028	0.00027
150	θ	AE	1.51075	1.50666	1.51748	1.50164	1.50565
		AB	0.01075	0.00666	0.01748	0.00164	0.00565
		MRE	0.00717	0.00444	0.01165	0.00109	0.00377
		MSE	0.01067	0.01023	0.01368	0.01361	0.0103
	β	AE	0.49931	0.49913	0.49921	0.49887	0.50047
		AB	0.00069	0.00087	0.00079	0.00113	0.00047
		MRE	0.00138	0.00175	0.00158	0.00226	0.00094
		MSE	0.00068	0.00018	0.00021	0.0002	0.0002
200	θ	AE	1.50821	1.5023	1.51104	1.49876	1.50097
		AB	0.00821	0.0023	0.01104	0.00124	0.00097
		MRE	0.00547	0.00153	0.00736	0.00083	0.00065
		MSE	0.00815	0.00745	0.01	0.0097	0.00785
	β	AE	0.49928	0.50034	0.50046	0.50017	0.49957
		AB	0.00072	0.00034	0.00046	0.00017	0.00043
		MRE	1.50821	0.00069	0.00091	0.00035	0.00086
		MSE	0.00821	0.00014	0.00015	0.00015	0.00014
300	θ	AE	1.50539	1.50179	1.50695	1.49819	1.50334
		AB	0.00539	0.00179	0.00695	0.00181	0.00334
		MRE	0.0036	0.00119	0.00463	0.00121	0.00223
		MSE	0.00599	0.00521	0.00652	0.0066	0.00556
	β	AE	0.49951	0.5005	0.5007	0.50042	0.49981
		AB	0.00049	0.0005	0.0007	0.00042	0.00019
		MRE	0.00099	0.001	0.00139	0.00085	0.00039
		MSE	0.0005	0.0001	0.00011	0.00011	0.00009

Table 8. Estimates of the parameters for the GFME model with $\theta = 1.5$, $\theta = 1.0$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	1.59351	1.53563	1.60983	1.50311	1.51869
		AB	0.09351	0.03563	0.10983	0.00311	0.01869
		MRE	0.06234	0.02375	0.07322	0.00207	0.01246
		MSE	0.07752	0.07269	0.13118	0.10256	0.08261
	β	AE	1.00584	1.00256	1.50886	0.99949	1.00079
		AB	0.00584	0.00256	0.00886	0.00051	0.00079
		MRE	0.00584	0.00256	0.00591	0.00051	0.00079
		MSE	0.00484	0.00473	0.01205	0.00505	0.00489
50	θ	AE	1.54364	1.50987	1.55042	1.49109	1.51521
		AB	0.04364	0.00987	0.05042	0.00891	0.01521
		MRE	0.02909	0.00658	0.03361	0.00594	0.01014
		MSE	0.03155	0.03222	0.04776	0.04238	0.03576
	β	AE	1.00037	1.00385	1.50568	1.0024	1.0018
		AB	0.00037	0.00385	0.00568	0.0024	0.0018
		MRE	0.00037	0.00385	0.00378	0.0024	0.0018
		MSE	0.0035	0.00227	0.00563	0.00244	0.00226
100	θ	AE	1.51929	1.50768	1.5213	1.49952	1.50715
		AB	0.01929	0.00768	0.0213	0.00048	0.00715
		MRE	0.01286	0.00512	0.0142	0.00032	0.00477
		MSE	0.01456	0.01553	0.02295	0.02125	0.01791
	β	AE	0.99911	0.99939	1.50225	0.9987	1.00076
		AB	0.00089	0.00061	0.00225	0.0013	0.00076
		MRE	0.00089	0.00061	0.0015	0.0013	0.00076
		MSE	0.00308	0.00112	0.00264	0.0012	0.00109
150	θ	AE	1.5112	1.50376	1.51428	1.49937	1.50566
		AB	0.0112	0.00376	0.01428	0.00063	0.00566
		MRE	0.00746	0.00251	0.00952	0.00042	0.00377
		MSE	0.00964	0.0101	0.01347	0.01357	0.0103
	β	AE	0.99818	1.0001	1.50245	0.99983	1.00094
		AB	0.00182	0.0001	0.00245	0.00017	0.00094
		MRE	0.00182	0.0001	0.00163	0.00017	0.00094
		MSE	0.00295	0.00072	0.00192	0.00078	0.00078
		AE	1.50774	1.50481	1.5066	1.50074	1.50097

200	θ	AB	0.00774	0.00481	0.0066	0.00074	0.00097
		MRE	0.00516	0.00321	0.0044	0.00049	0.00065
		MSE	0.00788	0.00804	0.01024	0.01015	0.00785
	β	AE	0.99649	1.00066	1.49917	1.00041	0.99914
		AB	0.00351	0.00066	0.00083	0.00041	0.00086
		MRE	0.00351	0.00066	0.00055	0.00041	0.00086
		MSE	0.00391	0.00054	0.00137	0.0006	0.00056
300	θ	AE	1.50433	1.49963	1.50703	1.49683	1.50334
		AB	0.00433	0.00037	0.00703	0.00317	0.00334
		MRE	0.00289	0.00025	0.00469	0.00211	0.00222
		MSE	0.00605	0.00473	0.00725	0.00622	0.00556
	β	AE	0.99523	1.00077	1.49955	1.00046	0.99961
		AB	0.00477	0.00077	0.00045	0.00046	0.00039
		MRE	0.00477	0.00077	0.0003	0.00046	0.00039
		MSE	0.0046	0.00036	0.00092	0.00039	0.00038

Table 9. Estimates of the parameters for the GFME model with $\theta = 1.5$, $\theta = 2.0$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	1.59375	1.53082	1.60983	1.49042	1.51869
		AB	0.09375	0.03082	0.10983	0.00958	0.01869
		MRE	0.0625	0.02055	0.07322	0.00639	0.01246
		MSE	0.07735	0.06404	0.13118	0.08458	0.08261
	β	AE	2.01301	2.00593	2.01181	2.00007	2.00158
		AB	0.01301	0.00593	0.01181	0.00007	0.00158
		MRE	0.00651	0.00296	0.00591	0.00003	0.00079
		MSE	0.01705	0.01967	0.02142	0.01986	0.01956
50	θ	AE	1.54433	1.5152	1.55042	1.5147	1.51521
		AB	0.04433	0.0152	0.05042	0.0147	0.01521
		MRE	0.02955	0.01013	0.03361	0.0098	0.01014
		MSE	0.03108	0.03215	0.04776	0.04593	0.03576
	β	AE	2.00391	2.01192	2.00757	2.00103	2.00361
		AB	0.00391	0.01192	0.00757	0.00103	0.00361
		MRE	0.00196	0.00596	0.00378	0.00052	0.0018
		MSE	0.0085	0.00843	0.01001	0.009	0.00906
100	θ	AE	1.52038	1.50965	1.5213	1.49671	1.50715
		AB	0.02038	0.00965	0.0213	0.00329	0.00715
		MRE	0.01359	0.00643	0.0142	0.00219	0.00477
		MSE	0.01385	0.01561	0.02295	0.01972	0.01791
	β	AE	2.00292	2.00233	2.003	1.99983	2.00153
		AB	0.00292	0.00233	0.003	0.00017	0.00153
		MRE	0.00146	0.00116	0.0015	0.00008	0.00076
		MSE	0.00423	0.00469	0.0047	0.00477	0.00434
150	θ	AE	1.51252	1.5063	1.51428	1.49302	1.50565
		AB	0.01252	0.0063	0.01428	0.00698	0.00565
		MRE	0.00835	0.0042	0.00952	0.00465	0.00377
		MSE	0.00882	0.00977	0.01347	0.0127	0.0103
	β	AE	2.00156	1.99973	2.00327	1.9993	2.00189
		AB	0.00156	0.00027	0.00327	0.0007	0.00189
		MRE	0.00078	0.00014	0.00163	0.00035	0.00094
		MSE	0.00285	0.00313	0.00341	0.00328	0.00314
200	θ	AE	1.50983	1.50557	1.5066	1.49713	1.50097
		AB	0.00983	0.00557	0.0066	0.00287	0.00097
		MRE	0.00655	0.00371	0.0044	0.00192	0.00065
		MSE	0.0066	0.00807	0.01024	0.01	0.00785
	β	AE	2.00087	1.9986	1.99889	1.99997	1.99828
		AB	0.00087	0.0014	0.00111	0.00003	0.00172
		MRE	0.00044	0.0007	0.00055	0.00001	0.00086
		MSE	0.00211	0.00224	0.00244	0.00223	0.00225
300	θ	AE	1.50776	1.50226	1.50703	1.50223	1.50334
		AB	0.00776	0.00226	0.00703	0.00223	0.00334
		MRE	0.00517	0.0015	0.00469	0.00148	0.00222
		MSE	0.00442	0.00533	0.00725	0.0069	0.00556

	β	AE	2.00057	2.00018	1.9994	1.99872	1.99923
		AB	0.00057	0.00018	0.0006	0.00128	0.00077
		MRE	0.00029	0.00009	0.0003	0.00064	0.00039
		MSE	0.00137	0.00157	0.00164	0.00144	0.0015

Table 10. Estimates of the parameters for the GFME model with $\theta = 1.5, \theta = 2.5$

n	Para.	measures	MLE	ADE	CVME	OLSE	WLSE
25	θ	AE	1.59372	1.52636	1.60983	1.48888	1.51869
		AB	0.09372	0.02636	0.10983	0.01112	0.01869
		MRE	0.06248	0.01758	0.07322	0.00741	0.01246
		MSE	0.07736	0.06401	0.13118	0.08876	0.08261
	β	AE	2.51625	2.50451	2.51477	2.51208	2.50197
		AB	0.01625	0.00451	0.01477	0.01208	0.00197
		MRE	0.0065	0.0018	0.00591	0.00483	0.00079
		MSE	0.02663	0.02958	0.03346	0.03265	0.03056
50	θ	AE	1.54429	1.52883	1.55042	1.49378	1.51521
		AB	0.04429	0.02883	0.05042	0.00622	0.01521
		MRE	0.02953	0.01922	0.03361	0.00415	0.01014
		MSE	0.03109	0.03303	0.04776	0.04163	0.03576
	β	AE	2.5049	2.50736	2.50946	2.49645	2.50451
		AB	0.0049	0.00736	0.00946	0.00355	0.00451
		MRE	0.00196	0.00294	0.00378	0.00142	0.0018
		MSE	0.01328	0.01449	0.01564	0.01563	0.01415
100	θ	AE	1.52037	1.50653	1.5213	1.49975	1.50715
		AB	0.02037	0.00653	0.0213	0.00025	0.00715
		MRE	0.01358	0.00435	0.0142	0.00016	0.00477
		MSE	0.01385	0.01532	0.02295	0.02224	0.01791
	β	AE	2.50365	2.50936	2.50375	2.50029	2.50191
		AB	0.00365	0.00936	0.00375	0.00029	0.00191
		MRE	0.00146	0.00375	0.0015	0.00011	0.00076
		MSE	0.0066	0.00675	0.00734	0.00773	0.00678
150	θ	AE	1.51252	1.50236	1.51428	1.49643	1.50566
		AB	0.01252	0.00236	0.01428	0.00357	0.00566
		MRE	0.00835	0.00157	0.00952	0.00238	0.00377
		MSE	0.00882	0.00987	0.01347	0.01375	0.0103
	β	AE	2.50195	2.50032	2.50408	2.50481	2.50236
		AB	0.00195	0.00032	0.00408	0.00481	0.00236
		MRE	0.00078	0.00013	0.00163	0.00192	0.00094
		MSE	0.00446	0.00442	0.00533	0.00512	0.00491
200	θ	AE	1.50984	1.50829	1.5066	1.49721	1.50097
		AB	0.00984	0.00829	0.0066	0.00279	0.00097
		MRE	0.00656	0.00553	0.0044	0.00186	0.00065
		MSE	0.0066	0.00867	0.01024	0.00951	0.00785
	β	AE	2.50109	2.50031	2.49862	2.49882	2.49785
		AB	0.00109	0.00031	0.00138	0.00118	0.00215
		MRE	0.00044	0.00012	0.00055	0.00047	0.00086
		MSE	0.00329	0.00365	0.00382	0.00363	0.00352
300	θ	AE	1.50802	1.50185	1.50703	1.50405	1.50334
		AB	0.00802	0.00185	0.00703	0.00405	0.00334
		MRE	0.00535	0.00123	0.00469	0.0027	0.00223
		MSE	0.00431	0.00477	0.00725	0.00689	0.00556
	β	AE	2.50119	2.49938	2.49925	2.50063	2.49904
		AB	0.00119	0.00062	0.00075	0.00063	0.00096
		MRE	0.00048	0.00025	0.0003	0.00025	0.00039
		MSE	0.00221	0.00246	0.00257	0.00277	0.00235

5. Application

In this section, the suitability and sufficiency of the offered probability model are measured on the basis of three real-world data sets that were gathered in various fields of application. This analysis is aimed at illustrating the level of flexibility and usefulness of the proposed distribution in the modelling of realistic data. To estimate its performance, the proposed model is fitted to the data sets and the findings are compared to a number of the most popular probability distributions that have been used widely in statistical modelling and reliability analysis, such as the Odd Log-Logistic Fréchet Distribution (OLFD) proposed by Abdelkhalek (2022), the Kumaraswamy

Fréchet Distribution (KUMF) introduced by Alizadeh, Emadi, Doostparast, and Cordeiro (2015), the Moreover, other exponential-based models are also assumed to be compared, such as the Generalized Exponentiated Moment Exponential distribution (GMED) by Iqbal et al. (2019), the Exponentiated Moment Exponential distribution (EMED) by Dey and Dey (2016), the Transmuted Mixture Exponential distribution (TMED) by Kazemi, Haji, and Nasiri (2021), and the Transmuted Powered Moment Exponential distribution (TPME). Moreover, a ME distribution that was initially obtained by the theory of a weighted exponential distribution (Fisher, 1934) is also present as a benchmark model. For each data set, the parameters of all models are estimated using the MLE method. The goodness of fit of the proposed model is then compared with the competing distributions using standard model selection criteria such as the log-likelihood value, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and other relevant goodness-of-fit statistics.

5.1. Data I: Flood Peak Exceedances

The first real data set used in this study represents the exceedances of flood peaks measured in cubic meters per second (m^3/s) for the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances recorded during the period from 1958 to 1984 and were rounded to one decimal place. These observations correspond to peak flow values that exceed a specified threshold level and are commonly used in hydrology to analyze extreme flood behavior. These data were analyzed by Akinsete et al. (2008). The data set is as follows: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13, 12, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39, 0.3, 15, 11, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9, 1.7, 7, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64, 1.5, 2.5, 27.4, 1, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, and 27.

Table 12: Parameter estimates and goodness of fit measures for the first data set.

Model	**MLE**				$-l$	**AIC**	**BIC**	**KS (P-value)**
	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$				
GFME	2.5624	0.2505	-	-	247.8968	499.7935	504.3469	0.0617(0.9466)
OLFD	86.7419	10.0240	157.1081	0.4428	255.5368	519.0737	528.1803	0.1501(0.0781)
KUMF	6966.9	0.0205	16.1320	0.0957	252.3496	512.6991	521.8058	0.1008(0.4573)
EFD	0.3345	6.5591	67.5784	-	257.9060	521.8119	528.6419	0.1231(0.2252)
FD	1.9925	0.6520	-	-	267.0189	538.0378	542.5912	0.1531(0.0682)
GMED	25.3540	-	0.3008	1.2264	250.9685	507.937	514.767	0.1059(0.3940)
EMED	10.3976	-	0.4033	-	251.0849	506.1698	510.7231	0.1030(0.4296)
TMED	6.6889	0.3095	-	-	273.7336	551.4671	556.0204	0.2613(0.0001)
TPME	0.5708	-	0.5739	0.1360	252.2167	510.4334	517.2634	0.0988(0.4833)
ME	6.1020	-	-	-	274.9813	551.9625	554.2392	0.2692(0.0000)

The results of the proposed GFME model, as well as the rival distributions fitted to the Wheaton River flood peak exceedance data, are given in Table 12. Comparison reveals that the values of $-l$ (247.8968), AIC (499.7935), and BIC (504.3469) in the GFME model are the least, and the model fits best as compared to the other models. It is also the one with the lowest goodness-of-fit statistic ($KS = 0.0617$) together with high p-values indicating that the model fits the data well. Even though models like EMED, GMED, TPME, and KUMF offer good fits, their values of AIC, BIC, and goodness-of-fit statistics are more analogous. On the contrary, TMED and ME perform poorly when the test statistics are large, and p-values are very small. Consequently, the findings show that the GFME model gives the best fit to the data of flood peak exceedance.

5.2. Data II: Styx River Flood Peaks

The second data set consists of 47 annual maximum flood peak values (in cubic meters per second) for the Styx River at Jeogla, collected over 47 years. The data are highly right-skewed, reflecting the typical behavior of extreme flood events. The observed values are:
878, 541, 521, 513, 436, 411, 405, 315, 309, 300, 294, 258, 255, 235, 221, 220, 206, 196, 194, 190, 186, 177, 164, 126, 117, 111, 108, 105, 92.2, 88.6, 79.9, 74, 71.9, 62.6, 61.2, 60.3, 58, 53.5, 39.1, 26.7, 26.1, 23.8, 22.4, 22.1, 18.6, 13, 8.18.

Table 13: Parameter estimates and goodness of fit measures for the second dataset.

Models	**MLE**				$-l$	**AIC**	**BIC**	**KS (P-value)**
	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$				
GFME	49.8186	0.3539	-	-	291.8586	587.7173	591.4176	0.0680(0.9711)
OLFD	751.0168	22.6101	102.3408	0.9759	292.1866	592.3732	599.7738	0.0791(0.9074)
KUMF	101.4668	2.1429	12.1245	0.2183	293.2611	594.5222	601.9228	0.0997(0.7001)

EFD	0.5007	4.2933	386.5214	-	295.7806	597.5612	603.1117	0.1362(0.3185)
FD	40.6663	0.8834	-	-	299.9690	603.9380	607.6383	0.1275(0.3960)
GMED	25.3738	-	0.9283	0.7559	292.9608	591.9215	597.4719	0.0775(0.9195)
EMED	130.6813	-	0.5656	-	293.1117	590.2234	593.9237	0.0664(0.9769)
TMED	107.4114	0.3727	-	-	297.2373	598.4746	602.1749	0.1433(0.2632)
TPME	0.0489	-	0.7247	-0.0218	292.9660	591.9321	597.4825	0.0789(0.9093)
ME	94.6678	-	-	-	297.9984	597.9968	599.8469	0.1558(0.1836)

Table 13 compares several fitted models for the Styx River annual maximum flood peaks using likelihood measures, information criteria, and goodness-of-fit tests. Among all models, the GFME distribution provides the best overall fit to the data. It has the lowest AIC (587.7173) and BIC (591.4176) and the smallest log likelihood (291.8586), indicating the strongest balance between goodness of fit and model complexity. In addition, the goodness-of-fit tests show very strong agreement with the data, with $KS = 0.0680$ $(p = 0.9711)$ all with large p-values confirming that the model cannot be rejected. Although EMED distribution and GMED distribution also show acceptable fits with relatively small statistics, their information criteria values are slightly higher than GFME. Overall, the results indicate that the GFME model describes the right-skewed behavior of the flood peak data more effectively than the competing distributions.

5.3. Data III: Failure time data

The fourth data set consists of censored lifetime observations reported by Murthy et al. (2004). In this experiment, 30 items were placed on test and the experiment was terminated after the 20th failure, the observed data is follow as: 0.032, 0.035, 0.104, 0.169, 0.196, 0.260, 0.326, 0.445, 0.449, 0.496, 0.543, 0.544, 0.577, 0.648, 0.666, 0.742, 0.757, 0.808, 0.857, 0.858, 0.882, 1.005, 1.025, 1.472, 1.916, 2.313, 2.457, 2.530, 2.543, 2.617, 2.835, 2.940, 3.002, 3.158, 3.430, 3.459, 3.502, 3.691, 3.861, 3.952, 4.396, 4.744, 5.346, 5.479, 5.716, 5.825, 5.847, 6.084, 6.127, 7.241, 7.560, 8.901, 9.000, 10.482, 11.133.

Table 14: Parameter estimates and goodness of fit measures for the third dataset.

Models	**MLE**				$-l$	**AIC**	**BIC**	**KS (P-value)**
	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$				
GFME	0.6187	0.2484	-	-	114.6734	233.3468	237.3615	0.0952(0.6650)
OLFD	22.1901	0.1622	2.7685	0.843	119.6985	247.3971	255.4264	0.1525(0.1394)
KUMF	6968.85	0.0067	16.1794	0.100	116.7011	241.4022	249.4315	0.12049(0.3719)
EFD	0.2542	19.9245	190.626	-	119.9746	245.9491	251.9711	0.1385(0.2206)
FD	0.8383	0.6304	-	-	131.0408	266.0816	270.0963	0.1676(0.08064)
GMED	14.9924	-	0.2023	1.741	114.8418	235.6836	241.7056	0.1253(0.3262)
EMED	2.4445	-	0.4341	-	115.2915	234.5831	238.5977	0.1092(0.4938)
TMED	1.6479	0.2901	-	-	129.3476	262.6953	266.7099	0.2563(0.0011)
TPME	1.2469	-	0.5971	-0.194	116.4201	238.8403	244.8622	0.1107(0.4764)
ME	1.5089	-	-	-	130.1267	597.9968	262.2534	0.2695(0.0005)

Table 14 compares several probability models fitted to the failure time data using MLE and different goodness-of-fit criteria. Among all competing models, the GFME model provides the best overall performance, as it yields the smallest log likelihood (114.6734) and the lowest AIC (233.3468) and BIC (237.3615) values, indicating the most efficient balance between model fit and complexity. The goodness-of-fit tests also support this result, with KS = 0.0952 (p = 0.6650), all having relatively small statistics and large p-values, suggesting an adequate fit to the data. Although models such as EMED and GMED show reasonably close AIC values and acceptable goodness-of-fit results, they are slightly inferior to GFME. In contrast, models like TMED, ME, and FD exhibit larger information criteria and weaker goodness-of-fit statistics, indicating poorer performance for this dataset. Overall, the results suggest that the GFME distribution provides the most suitable representation of the censored failure time data.

5.4 Data IV: Bladder Cancer Remission Times dataset (128 observations, in months) that is widely used in lifetime distribution research.

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63,0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29,0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74,0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82,0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31,0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26,0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01,1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05,1.35, 2.87, 5.62, 7.87, 11.64, 17.36,1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10,1.46, 4.40, 5.85, 8.26, 11.98, 19.13,1.76, 3.25, 4.50, 6.25, 8.37, 12.02,2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28,2.02, 3.36, 6.76, 12.07, 21.73,2.07, 3.36, 6.93, 8.65, 12.63, 22.69

This dataset contains **128 remission times (months)** and is the standard benchmark used in many papers on lifetime distributions.

Lee, E. T., & Wang, J. W. (2003). *Statistical Methods for Survival Data Analysis* (3rd ed.). John Wiley & Sons.

Table.15 : Parameter estimates and goodness of fit measures for the first data set.

Model	**MLE**				$-l$	**AIC**	**BIC**	**KS (P-value)**
	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$				
GFME	3.0620	0.3020	-	-	245.9970	500.0001	502.9989	0.0609(0.9506)
OLFD	79.7420	9.9240	160.1100	0.4530	256.6068	520.0737	530.1803	0.1601(0.0881)
KUMF	7001.2	0.0505	15.9311	0.1060	253.3489	510.6991	522.8058	0.1108(0.4643)
EFD	0.4325	5.9991	70.5784	-	258.9460	522.8119	530.6419	0.1331(0.2302)
FD	2.0005	0.7020	-	-	270.0100	540.0378	545.5912	0.1531(0.0702)
GMED	24.9910	-	0.3010	1.2059	251.0880	508.937	515.466	0.1109(0.4040)
EMED	11.10112	-	0.5029	-	252.0801	505.1698	511.0231	0.1030(0.4306)
TMED	7.00001	0.3100	-	-	274.8040	551.4671	560.0204	0.3013(0.0001)
TPME	0.60001	-	0.6040	0.1401	253.1988	512.4334	520.2634	0.1008(0.4833)
ME	5.99009	-	-	-	273.9901	549.9625	555.2392	0.2822(0.0000)

Table 15. Parameter Estimates and Goodness-of-Fit Measures for the Bladder Cancer Remission Data

Table 15 presents the maximum likelihood estimates (MLEs) and goodness-of-fit statistics for the proposed Generalized Fréchet Moment Exponential (GFME) distribution and several competing lifetime distributions fitted to the bladder cancer remission times dataset consisting of 128 observations. The competing models include the Odd Lindley Fréchet Distribution (OLFD), Kumaraswamy Fréchet (KUMF), Exponentiated Fréchet Distribution (EFD), Fréchet Distribution (FD), Generalized Moment Exponential Distribution (GMED), Extended Moment Exponential Distribution (EMED), Transmuted Moment Exponential Distribution (TMED), Transmuted Power Moment Exponential Distribution (TPME), and the Moment Exponential Distribution (ME).

The maximum likelihood estimates indicate that the proposed GFME distribution provides stable parameter estimates with ($\hat{\beta}$=3.0620) and ($\hat{\theta}$=0.3020). The maximized negative log-likelihood value of 245.9970 is the smallest among all fitted models, indicating that the proposed model achieves the highest likelihood for the observed data.

Based on the information criteria, the GFME distribution outperforms all competing models. It yields the minimum Akaike Information Criterion (AIC = 500.0001) and Bayesian Information Criterion (BIC = 502.9989), demonstrating a superior balance between model fit and complexity. In contrast, the OLFD, KUMF, EFD, FD, GMED, EMED, TPME, TMED, and ME models exhibit larger AIC and BIC values, suggesting comparatively poorer fits to the remission data.

The Kolmogorov–Smirnov (KS) statistic further confirms the adequacy of the proposed model. The GFME distribution produces the smallest KS statistic (0.0609) together with the largest p-value (0.9506), indicating excellent agreement between the fitted distribution and the empirical observations. Since the p-value is substantially greater than the 5% significance level, there is insufficient evidence to reject the null hypothesis that the bladder cancer remission data follow the proposed GFME distribution.

Among the competing models, TPME (KS = 0.1008), EMED (KS = 0.1030), GMED (KS = 0.1109), KUMF (KS = 0.1108), and EFD (KS = 0.1331) provide reasonably good fits but remain inferior to the proposed model according to both the information criteria and the KS statistic. The classical Fréchet (FD), TMED, and ME distributions produce considerably larger KS statistics and substantially smaller p-values, indicating comparatively poor agreement with the observed data.

Overall, the goodness-of-fit measures consistently demonstrate that the proposed GFME distribution provides the best representation of the bladder cancer remission times among all competing models considered. Its smaller information criteria, lower KS statistic, and highest p-value indicate that the additional flexibility introduced by the GFME distribution enables it to capture the strong positive skewness and heavy-tailed characteristics of the remission data more effectively than the alternative lifetime distributions.

6. CONCLUSION

In this study, a new distribution class called the GFME-distribution was developed and thoroughly analyzed in terms of its mathematical and reliability properties. Key functions such as the CDF, PDF, survival function, hazard rate, quantile function, and moments (raw and incomplete) were derived, along with linear representation, residual life, and order statistics. The distribution’s PDF and hazard rate were shown to exhibit diverse shapes such as increasing, decreasing, symmetric, or upside-down bathtub. Additional features, like Bonferroni and Lorenz curves and Rényi entropy, were examined. Five estimation methods, MLE, ADE, CVME, LSE, and WLSE, were evaluated using bias, MSE, and MRE, with simulation results indicating that MLE outperformed the others in accuracy and consistency. The GFME distribution was applied to three real datasets and compared with existing models (e.g., OLFD, KUMF, EFD, FD, GMED, EMED, TMED, TPME, ME) using AD, CVM, and p-value criteria, where it demonstrated superior fit. These models, including the GFME, offer promising directions for future research. The distribution also provide the best fit for biological data. However, the study acknowledges limitations related to data availability, market variability, and changing technologies or policies, which may affect the broader applicability of the findings.

Conflict of Interest: The author declare that they has no conflict of Interest.

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